DYNAMICS OF CHARGED PARTICLES

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by

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PREFACE

This book contains a study of the dynamics of ionized matter in electric and magnetic fields, mainly on the basis of a single particle picture. The treatment starts with the fundamental equations of classical theory and leads from this level up to the present state of research. With a few exceptions it should give a complete line of deductions all the way to the final results. Much space has been devoted to clarify conceptual difficulties, and specific problems are often considered from different angles of approach. The results are illuminated by simple physical interpretations. The deductions have been presented in a way to emphasize the logical relationship between chapters and to tie together related results.

With the exception of some minor changes and addenda the contents of this volume was completed in June 1962. It mainly includes contributions which were known by the author up to the same date.

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B. L.

Royal Institute of Technology
Stockholm, October 1963

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INTRODUCTION

1. Historical Survey

The laws of classical mechanics and of classical theory on electricity and magnetism were established already at the times of Newton and Maxwell. Applications of the laws to the motion of charged particles also attracted an early interest in several fields of research. In 1907 Störmer started a study of the orbits of charged particles in a magnetic dipole field. His results have later been applied in the exploration of cosmic rays and their orbits in the terrestrial magnetic field. Classical theory was also used in the earlier attempts by THOMSON [1910], RUTHERFORD [1911], BOHR [1913] and others to describe the atomic structure. Among the great variety of technical applications to the motion of charged particles may be mentioned the design and study of electronic tubes. Practical developments of such tubes began already in the 1890's, but a complete electron optical theory was first formulated by BUSCH [1926].

In 1908 Hale discovered that the sunspots are associated with strong magnetic fields. Since this time astronomers and astrophysicists have shown an ever increasing interest in the behaviour of ionized matter in a magnetic field. In 1937 Ferraro found that the layers of a magnetized and ionized body will be forced to rotate with the same angular speed at every point on a magnetic field line. This "isorotation law" illustrates one of the fundamental mechanisms by which charged particles interact with a magnetic field. The importance of this became even more clear when ALFVÉN [1942] showed that the magnetic field lines act like elastic strings to which ionized matter is "frozen". As a consequence, magnetohydrodynamic waves can propagate along the magnetic lines of force. After Alfvén's discovery of these waves an almost explosive development started in the research on magnetized and electrically conducting media. The theoretical approach has followed both the lines of a single particle picture and those of a macroscopic fluid model. To a great extent the former has been based upon the perturbation theory on particle orbits, as originally developed by ALFVÉN [1940]. Important con-

tributions to this approach are also due to CHAPMAN and COWLING [1939, 1952], SPITZER [1952] and many others.

During the last ten years the problem of controlled thermonuclear fusion has added a new and important application to the physics of ionized matter. Its basic problems are closely related to those of cosmical physics. They concern the orbits of charged particles in electric and magnetic fields, the determination of forbidden regions of the particle motion, diffusion of a plasma across a magnetic field, wave phenomena, plasma stability, and radiation problems.

A recent application also involves the design of magnetohydrodynamic energy converters. In such devices thermal energy of an ionized gas is transformed directly into electric energy. The latter is generated by electric currents which are induced during an expansion of the gas across a magnetic field.

The motion of charged particles in electric and magnetic fields has earlier been surveyed in a number of monographs and reviews such as those by CHAPMAN and COWLING [1939], ALFVÉN [1950], SPITZER [1956], ALLIS [1956], CHANDRASEKHAR [1958], DUNGEY [1958], SIMON [1959], DELCROIX [1960], LINHART [1960], CHANDRASEKHAR and TREHAN [1960], ROSE and CLARK [1961], NORTHROP [1961], FERRARO and PLUMPTON [1961], FÜNFER and LEHNER [1962], THOMPSON [1962], ALFVÉN and FÄLTHAMMAR [1963] and LONGMIRE [1963].

2. Dynamics of an Ionized Gas

The dynamics of an ionized gas is governed by basic equations which have been known in detail since the end of the last century. Nevertheless, the final solution is still missing for many important applications to theory. The reason for this is that several types of interactions may take place simultaneously between ionized matter and electric and magnetic fields. The richness in combinations between such interactions creates a system of high complexity.

As a first step in a theoretical study of an ionized gas one may start from the motion of a single charged particle in electric and magnetic fields which are considered as given constraints. The particle orbit can then be calculated in detail from the equation of motion, at least in principle. In most cases of interest, however, these calculations become complicated from the mathematical point of view and the result is unsurveyable. Further studies of the problem are still possible by making use of a perturbation method which

gives an approximate solution to the mean particle orbit. It describes the motion as that of a guiding centre around which the actual particle is gyrating. The perturbation theory contains a number of approximate constants of the motion. It also provides convenient tools for the calculation of the particle orbit and simplifies its physical interpretation.

However, care is necessary when the perturbation theory is used to describe the dynamics of an ionized gas. It has to be kept in mind that the particle orbits are made up of a sum of two motions; the drift of the guiding centre and the superimposed gyration. Therefore, the real flux of matter, momentum and energy is not solely determined by the motion of the guiding centra but also by the gyration. The simple fact that the particles gyrate in circle-like orbits leads to results which at a first sight may seem to be quite peculiar. Thus, there can exist a net flux of particles in absence of any guiding centre motion, and the reverse may also be true in some cases. A thorough examination of the complete particle motion actually shows that these statements are correct.

Our next step is to consider an ionized gas at the large particle densities which are actual in most physical applications. Under such conditions even the slightest separation between ions and electrons gives rise to considerable electric space charges. It is then no longer permissible to consider the electric field as a given constraint which is determined only by external sources. Instead, the electric field has to be modified by a contribution from the separated space charges. These charges are, on the other hand, very sensitive to changes in the particle orbits. Therefore, the orbits have to be calculated with a high degree of accuracy to give the right values of the space charges and of the electric field. This also implies that the lowest order approximations of the orbit theory may lead to erroneous results when the transition is made from the single particle picture to that of an ionized gas.

The motion of surfaces of constant particle density sometimes produces additional charge separation effects. It will take place at a velocity which differs both from the mean particle velocity and from the velocity of the guiding centra. In certain cases this gives rise to an unequal compression or expansion of the ion and electron distributions in space. In combination with particle drifts across the magnetic field this effect leads to a space charge formation which changes the electric fields and the particle orbits.

The difficulties which arise in a treatment of the various effects just mentioned can be overcome in a correct development of the perturbation theory. The approximations then have to be carried out far enough both for the

guiding centre drift and for the gyration, and the latter have to be superimposed in an accurate way. To some extent such problems can be avoided in a macroscopic approach which starts from mean values deduced from Boltzmann's equation. However, another problem arises in such an approach. It lies in the determination of the pressure tensor. Whether the orbit theory or the macroscopic theory should be of greatest advantage depends upon the particular situation with which one may be faced. Also a direct solution of the Boltzmann equation is sometimes preferable.

Finally, when the induced electric currents in the ionized gas itself are no longer negligible, the magnetic field cannot be treated as a given constraint. In addition to the equation of motion the particles will then be coupled to the electromagnetic field by the induction law. In a general situation there is also a coupling between the particles and an electromagnetic radiation field.

3. Relationship between Chapters

In this volume the motion and confinement of charged particles in electric and magnetic fields is discussed, mainly on the basis of a single particle picture. Only a minor space has been devoted to descriptions in terms of a macroscopic fluid model. The particle picture and the fluid model are in many respects equivalent methods of approach. Some of the more involved problems have been treated by both methods. This has been done in order to illuminate the results from different angles and to clarify difficulties in their interpretation.

The relationship between chapters is demonstrated by Figure 1.1. The starting-points of the theory are summarized in chapters indicated by heavy frames. They lead to results of a general character which are developed in chapters marked by double frames. The applications of the general theories are then deduced in chapters marked by single frames. Arrows indicate the logical relationship between the chapters. Consequently, results deduced in an associated chapter are built on the contents of chapters from which the arrows originate. Attempt has been made to present an unbroken chain of deductions and to avoid repeated formulation of basic results.

From the basic equations of Chapter 2 follows the main line of approach. It is represented by the orbit theory of single particle motion in Chapter 3, the associated adiabatic invariants of Chapter 4, and the applications to magnetic compression phenomena, confinement of charged particles, and stability in Chapters 6, 7 and 8. The macroscopic theory has been developed

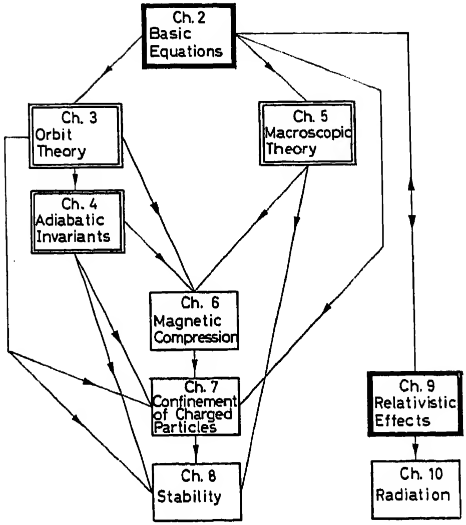


Fig. 1.1. Logical relationship between the chapters of the present volume. Heavy frames denote chapters which contain the starting-points of the theories. Double frames indicate chapters with results of general character, and single frames are used for applications to the general theories.

in Chapter 5 from the basic equations. It is used to complete Chapters 6, 7 and 8 and to give a deeper insight into the applications of these chapters. Part of the discussion on confinement in Chapter 7 also originates directly from the basic equations of Chapter 2. Finally, the same equations can be extended to include relativistic effects as demonstrated in Chapter 9. These results are applied to radiation problems in Chapter 10. They also react back through the basic equations of Chapter 2 and lead to additional conclusions about the problems of Chapters 3, 4, 6, 7 and 8.

During a first study of this book the reader may leave out paragraphs which are marked with an asterisk. It is hoped that the list of commonly used expressions as well as the subject index at the end of the volume may simplify the reading. All formulae are given in MKSA-units.

BASIC EQUATIONS

This chapter contains a review of the basic theories of the electromagnetic field and of classical mechanics. It does not make any claim to give a comprehensive description of these subjects. Only such parts will be treated which are directly connected with the problems of subsequent chapters. Whenever more rigorous and complete discussions are needed reference should be made to monographs such as those by STRATTON [1941], HEITLER [1954], MORSE and FESHBACH [1953] and GOLDSTEIN [1957].

1. The Electromagnetic Field

Throughout this volume we shall consider particles which represent electric point charges moving in vacuo, under the influence of an electric field $\boldsymbol{E}$ and a magnetic field $\boldsymbol{B}$. The fields are partly generated by external sources, partly by the point charges themselves. An explicit study of the microfields from individual charges will only be undertaken in some special applications to collision phenomena. In the main parts of this volume we substitute the actual electric and magnetic fields by their average values taken over macroscopically small regions in space and intervals of time. An approach of this kind seems to be justified in a plasma when the mean distance between particles is large enough for quantum mechanical effects to be neglected. However, it is not immediately clear that it should apply also to solids, e.g. in a theory on the motion of electrons in metals or on the magnetic properties of iron. Even in the classical limit the interpretation of the mean field has not yet become quite clear, because of the long range of the local electric and magnetic fields which arise from moving ions and electrons.

1.1. MAXWELL'S EQUATIONS

The electromagnetic field is governed by Maxwell's equations, where

$$\operatorname{curl} \boldsymbol{E} = - \partial \boldsymbol{B}/\partial t \tag{2.1}$$

is the law of electromagnetic induction and

$$\operatorname{curl} \boldsymbol{B}/\mu_0 = \boldsymbol{j} + \varepsilon_0 \, \partial \boldsymbol{E}/\partial t \tag{2.2}$$

expresses the magnetic field in terms of its sources. These are the electric

current density $\boldsymbol{j}$ due to conduction by moving charges and the displacement current given by the last term of (2.2). The magnetic permeability and the dielectric constant in vacuo are denoted by μ_0 and ε_0. In the definition of the current density the number of particles in a macroscopic element of volume is assumed to be very large and $\boldsymbol{j}$ is given by the mean flux of charge per unit area. Likewise, we also define an electric charge σ per unit volume and the condition for its conservation becomes:

$$\operatorname{div} \boldsymbol{j} = - \partial\sigma/\partial t \ . \tag{2.3}$$

Since a commutation of the operators ∇ and $\partial/\partial t$ is admissible a divergence operation on equations (2.1) and (2.2) gives, after combination with (2.3),

$$\frac{\partial}{\partial t} \operatorname{div} \boldsymbol{B} = 0 \tag{2.4}$$

and

$$\frac{\partial}{\partial t} (\varepsilon_0 \operatorname{div} \boldsymbol{E} - \sigma) = 0 \ . \tag{2.5}$$

If we now assume that the fields vanish at some time in their past the result becomes

$$\operatorname{div} \boldsymbol{B} = 0 \tag{2.6}$$

and

$$\operatorname{div} \boldsymbol{E} = \sigma/\varepsilon_0 \ . \tag{2.7}$$

Equations (2.2), (2.5) and (2.7) apply to conditions in vacuo and have to be modified when the electric and magnetic behaviour of dense matter is concerned. This can be done in a phenomenological approach where electric and magnetic polarization phenomena are represented by electric and magnetic susceptibilities. The latter are then considered as macroscopic properties of matter.

An analogous treatment is also possible in the case of an ionized gas, but does not always become necessary. In particular, it should be stressed that *all* electric currents and charges are taken explicitly into account in (2.2), (2.3) and (2.7). By combination of these equations with the equations of motion of an ionized gas one can study the polarization phenomena directly. This does not require any equivalent susceptibilities, dielectric constants and magnetic permeabilities to be introduced.

Should one in any case prefer to adopt these concepts, they also have to be

used with care. Sometimes there will exist effects in an ionized gas which cannot be expressed in terms of such equivalent parameters only. In order to avoid pitfalls it is therefore advisable to examine every particular case separately. This will become clear from the more detailed analysis in Ch. 3, § 2, and Ch. 8, §§ 2.4 and 2.5.

The condition (2.6) implies that $\boldsymbol{B}$ can always be represented as the curl of a vector potential $\boldsymbol{A}$:

$$\boldsymbol{B} = \operatorname{curl} \boldsymbol{A} \ . \tag{2.8}$$

Equation (2.1) then becomes

$$\operatorname{curl}\left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}\right) = 0 \tag{2.9}$$

or

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} \ , \tag{2.10}$$

where ϕ is the electric scalar potential. Substitution of expressions (2.8) and (2.10) into (2.2) yields

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla\left(\operatorname{div} \boldsymbol{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right) = -\mu_0 \boldsymbol{j} \ , \tag{2.11}$$

where $c = (\mu_0 \varepsilon_0)^{-\frac{1}{2}}$ is the velocity of light. Further, combination of (2.10) and (2.7) gives

$$\nabla^2 \phi + \operatorname{div} \frac{\partial \boldsymbol{A}}{\partial t} = -\sigma/\varepsilon_0 \ . \tag{2.12}$$

Up to this point the vector $\boldsymbol{A}$ has not been uniquely determined by the field $\boldsymbol{B}$ since it is always possible to add the gradient $\nabla\chi$ of an arbitrary scalar function χ to $\boldsymbol{A}$ in (2.8). Therefore the field

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\chi \tag{2.13}$$

also satisfies (2.8). In order that the electric field shall also remain unchanged equation (2.10) requires that the electric potential now becomes

$$\phi' = \phi - \frac{\partial \chi}{\partial t} \ . \tag{2.14}$$

The substitution of the potentials $\boldsymbol{A}$, ϕ by $\boldsymbol{A}'$, ϕ' is called a *gauge transformation.* A special gauge of interest is that for which

$$\operatorname{div} \boldsymbol{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = \operatorname{div} \boldsymbol{A}' + \frac{1}{c^2}\frac{\partial \phi'}{\partial t} - \nabla^2\chi + \frac{1}{c^2}\frac{\partial^2 \chi}{\partial t^2} \tag{2.15}$$

vanishes. Since χ is arbitrary this is always possible by a proper choice of χ and the result is the Lorentz condition:

$$\operatorname{div} \boldsymbol{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0 \ . \tag{2.16}$$

By means of this condition equations (2.11) and (2.12) reduce to the inhomogeneous wave equations

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2} \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = - \mu_0 \boldsymbol{j} \ , \tag{2.17}$$

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = - \sigma / \varepsilon_0 \ . \tag{2.18}$$

The general solution of (2.17) and (2.18) consists of the general solution of the homogeneous equations with zero right hand members together with the special solutions

$$\boldsymbol{A} (\boldsymbol{\rho}, t) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{j} (\boldsymbol{\rho}^*, t - R^*/c)}{R^*} \, \mathrm{d} V^* \tag{2.19}$$

and

$$\phi (\boldsymbol{\rho}, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\sigma (\boldsymbol{\rho}^*, t - R^*/c)}{R^*} \, \mathrm{d} V^* \ . \tag{2.20}$$

Here $\boldsymbol{\rho}$ is the position vector of a certain field point, $\boldsymbol{\rho}^*$ is the position vector of the current and charge distributions, and $\boldsymbol{R}^* = \boldsymbol{\rho} - \boldsymbol{\rho}^*$. The integration has to be performed at the retarded time $t^* = t - R^*/c$ over the volume elements $\mathrm{d}V^*$ of the current and charge distributions. The solutions (2.19) and (2.20) are called the *retarded potentials*. They also satisfy (2.16).

1.2. MAGNETIC FIELD COORDINATES

In a study of the particle motion in a magnetic field it is sometimes useful to describe the latter in terms of two scalar functions, α and β, as introduced by JACOBI [1844] and STÖRMER [1916] and later used by SWEET [1950], LUNDQUIST [1952], and GRAD and RUBIN [1958]. We can write

$$\boldsymbol{A} = \alpha \nabla \beta + \nabla \chi \tag{2.21}$$

for any vector field $\boldsymbol{A}$. The magnetic field is then determined from (2.8) which yields

$$\boldsymbol{B} = \nabla \alpha \times \nabla \beta \ . \tag{2.22}$$

Both $\nabla\alpha$ and $\nabla\beta$ must be perpendicular to $\boldsymbol{B}$ as soon as they differ from zero. The divergence of the field determined by this equation is easily seen to vanish by means of well-known vector operations. Clearly the magnetic field lines are generated by the intersections between the surfaces $\alpha =$ const. and $\beta =$ const. and every field line can be specified by a certain pair of values α, β.

In addition, we introduce the coordinate s measuring the arc length along a certain line of force. The quantities (s, α, β) specify a point with respect to the magnetic field and can be used as curvilinear coordinates. These are often the "natural" coordinates of a problem where the magnetic field plays a fundamental rôle. It should be kept in mind that the coordinates themselves will change in time when the magnetic field does so and that $\partial\alpha/\partial t$ and $\partial\beta/\partial t$ are not necessarily constant on a field line since the field may change non-uniformly in space and time.

With the special gauge for which $\chi = 0$ substitution of the form (2.21) into equation (2.10) for the electric field gives, after partial derivation,

$$\boldsymbol{E} = -\nabla\left(\phi + \alpha\frac{\partial\beta}{\partial t}\right) + \frac{\partial\beta}{\partial t}\,\nabla\alpha - \frac{\partial\alpha}{\partial t}\nabla\beta\,. \tag{2.23}$$

In particular, the longitudinal component of the electric field becomes

$$E_{\parallel} = -\frac{\partial}{\partial s}\left(\phi + \alpha\frac{\partial\beta}{\partial t}\right) \tag{2.24}$$

since α and β are constant on a field line according to (2.22).

1.3. THE CONCEPT OF MAGNETIC FIELD LINES

In the preceding paragraph we have seen that the magnetic field can be described by two scalar functions, α and β, which generate a family of field lines filling the entire space. Up till now we have not specified what is meant by the motion of a magnetic field line. To illustrate the difficulties which arise in this connexion we apply some physical arguments to a simple example.

An electric generator consists of a closed circuit, parts of which are formed by conductors moving across a magnetic field. The source of the induced electric current can be pictured as an electromotive force. The latter is given by the rate at which the magnetic field lines are cut by the moving conductors. However, the concept of a magnetic field line and its motion with respect to matter is not uniquely determined. To illustrate this we consider

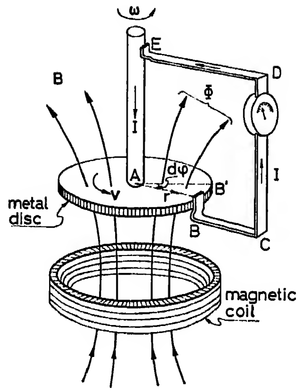


Fig. 2.1. Unipolar inductor, consisting of a metal disc rotating at angular velocity ω and connected to a fixed external circuit BCDE (cf. ALFVÉN [1950]).

the unipolar inductor of Figure 2.1 and follow an earlier discussion by ALFVÉN [1950]. The inductor consists of a magnetic coil which produces an axially symmetric magnetic field $\boldsymbol{B}$ in which a metal disc rotates around the axis of symmetry at an angular velocity ω. A fixed branch BCDE is electrically connected to the periphery of the disc at B and to its axis at E. The system works as a simple electric generator which produces a current I in the circuit ABCDEA.

There are at least two ways in which we can look upon the current generation in the present arrangement. We can assume that the magnetic field lines are fixed in space. Then, they are cut by a moving conductor only along the part AB of the circuit and the induced electromotive force becomes

$$\phi_{\mathrm{AB}} = \oint\limits_{\mathrm{ABCDEA}} (\boldsymbol{v} \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{l} = \int_{\mathrm{A}}^{\mathrm{B}} vB\mathrm{d}r = (\omega/2\pi)\, \Phi_{\mathrm{AB}} \,, \qquad (2.25)$$

where Φ_{AB} is the magnetic flux passing through the metal disc.

However, we can also consider all field lines to move with the disc at an angular velocity ω. Then, the part BCDE of the circuit will cut the field lines and produce an electromotive force which is obtained from an integral along BCDE:

$$\phi_{\mathrm{BA}} = -\int_{\mathrm{B}}^{\mathrm{A}} \omega r B\mathrm{d}r = (\omega/2\pi)\, \Phi_{\mathrm{AB}} = \phi_{\mathrm{AB}} \,. \qquad (2.26)$$

This gives the same result as when the field lines are assumed to be "at rest".

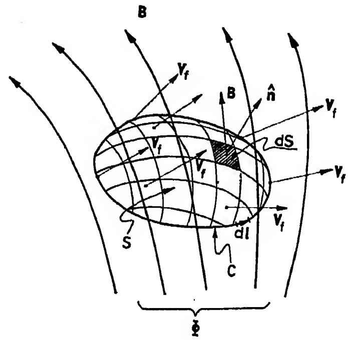


Fig. 2.2. The curve C and the enclosed surface S are carried along at all points with the local speed of an arbitrary velocity field $\boldsymbol{V}_f$. The flux Φ of the magnetic field $\boldsymbol{B}$ is enclosed by C.

The same result would be obtained for a circuit made of elastic conductors which are deformed from a shape given by ABCDEA to that given by AB′BCDEA in a time $\mathrm{d}t = \mathrm{d}\varphi/\omega$. The change in flux then becomes $\mathrm{d}\Phi/\mathrm{d}t = \omega\Phi_{AB}/2\pi$ per unit time.

Finally, observe that no net electromotive force and no current will be induced around the circuit if the external part BCDE is fixed to the rotating disc and to its axis such as to take part in the rotation. The change of the enclosed magnetic flux is zero in this case as well as the line integral of $\boldsymbol{v} \times \boldsymbol{B}$ around the circuit.

From this example we see that it is not possible or even essential to define a motion of the magnetic field lines; what is important is instead the *change of magnetic flux* inside the actual circuit. On the other hand, the picture of moving magnetic field lines may sometimes give a good illustration to the physics of a magnetized medium, provided that this picture is not taken too literally.

For a detailed analysis of the present questions we associate an arbitrary velocity field $\boldsymbol{V}_f$ with every point in the space occupied by a magnetic field $\boldsymbol{B}$ as indicated in Figure 2.2. Consider an arbitrary, closed curve C which moves with the local velocity $\boldsymbol{V}_f$ at all points of its perimeter, i.e., the curve is "carried along" by the velocity field. Also the elements $\mathrm{d}S$ of the enclosed surface S are assumed to move at this velocity. Among all possible fields $\boldsymbol{V}_f$ we specify two classes by definitions earlier given by NEWCOMB [1958]:

(i) The field $\boldsymbol{V}_f$ is *flux-preserving* if the magnetic flux enclosed by C remains constant during the motion.

(ii) The field $\boldsymbol{V}_f$ is *line-preserving* if field lines are transformed into field lines by the velocity $\boldsymbol{V}_f$. The definition is specified more in detail if a line element $d\boldsymbol{l}_0$ is considered which coincides with a certain magnetic field line at the time t_0. Line-preservation then implies that $d\boldsymbol{l}_0$ is carried along by $\boldsymbol{V}_f$ to new positions $d\boldsymbol{l}$ such as to coincide with other field lines at all later times t (Fig. 2.3).

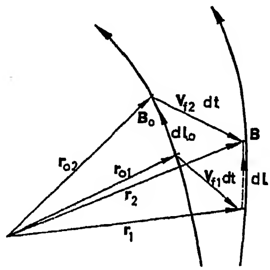


Fig. 2.3. A line element $d\boldsymbol{l}_0$ coinciding with the magnetic field $\boldsymbol{B}_0$ is carried along by the velocity $\boldsymbol{V}_f$ to a new position $d\boldsymbol{l}$ where it coincides with the corresponding field vector $\boldsymbol{B}$.

We first discuss flux-preservation in connexion with Figure 2.2. The change of flux per unit time of any vector field $\boldsymbol{B}$ due to the motion $\boldsymbol{V}_f$ of the contour C and its enclosed surface S is

$$\frac{d\Phi}{dt} = \int_S\int \hat{\boldsymbol{n}} \cdot \left[\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{V}_f \operatorname{div} \boldsymbol{B} + \operatorname{curl}(\boldsymbol{B} \times \boldsymbol{V}_f)\right] dS. \quad (2.27)$$

Here the integral is taken over the surface S and $\hat{\boldsymbol{n}}$ denotes the unit vector perpendicular to a surface element dS and pointing in the positive direction of circulation around C. The first contribution to the integral (2.27) is due to the change of the magnetic field strength at a fixed point in space. The second contribution arises from the motion of the surface S when it sweeps over the sources of the vector field $\boldsymbol{B}$ in space. Since $\boldsymbol{B}$ is a magnetic field in this particular case, we have $\operatorname{div} \boldsymbol{B} = 0$ and this contribution to the changes in flux will vanish. The last contribution to the integral is produced by the displacement and the deformation of the contour C due to the velocity $\boldsymbol{V}_f$. This contribution can arise from inhomogeneities in $\boldsymbol{V}_f$ as well as in $\boldsymbol{B}$.

Substitution of the law (2.1) of electromagnetic induction into (2.27) yields

$$\frac{d\Phi}{dt} = -\int_S\int \hat{\boldsymbol{n}} \cdot \operatorname{curl}(\boldsymbol{E} + \boldsymbol{V}_f \times \boldsymbol{B})\, dS = -\oint_C (\boldsymbol{E} + \boldsymbol{V}_f \times \boldsymbol{B}) \cdot d\boldsymbol{l}. \quad (2.28)$$

The condition for flux-preservation implies that the change in flux should become zero for all contours C and surfaces S. Therefore, the integrand of the surface integral must vanish identically. Hence the necessary and sufficient condition for flux-preservation becomes

$$\boldsymbol{E} + \boldsymbol{V}_{\mathrm{f}} \times \boldsymbol{B} = - \nabla \chi \tag{2.29}$$

where χ is a scalar field.

Now turn to the problem of line-preservation. Consider a line element $\mathrm{d}\boldsymbol{l}_0 = \boldsymbol{r}_{02} - \boldsymbol{r}_{01}$ which coincides with the magnetic field $\boldsymbol{B}_0$ at the time t_0 as shown in Figure 2.3. After a time $\mathrm{d}t$ it has been carried along by the velocity $\boldsymbol{V}_{\mathrm{f}}$ to a new position where it becomes

$$\mathrm{d}\boldsymbol{l} = \boldsymbol{r}_{02} + \boldsymbol{V}_{\mathrm{f}2}\mathrm{d}t - \boldsymbol{r}_{01} - \boldsymbol{V}_{\mathrm{f}1}\mathrm{d}t = \mathrm{d}\boldsymbol{l}_0 + (\mathrm{d}\boldsymbol{l}_0 \cdot \nabla)\, \boldsymbol{V}_{\mathrm{f}}\mathrm{d}t. \tag{2.30}$$

An observer who follows this motion measures a first order change in magnetic field strength

$$\boldsymbol{B} - \boldsymbol{B}_0 = (\boldsymbol{V}_{\mathrm{f}} \cdot \nabla)\, \boldsymbol{B}_0\mathrm{d}t + \frac{\partial}{\partial t}\, \boldsymbol{B}_0\mathrm{d}t. \tag{2.31}$$

The first term of the right hand member of this relation is due to the motion across the inhomogeneous magnetic field. The last term is produced by the field change at a fixed point in space. From (2.30) and (2.31) we obtain

$$\boldsymbol{B} \times \mathrm{d}\boldsymbol{l} = \boldsymbol{B}_0 \times [(\mathrm{d}\boldsymbol{l}_0 \cdot \nabla)\, \boldsymbol{V}_{\mathrm{f}}]\ \mathrm{d}t + \left[\left(\frac{\partial}{\partial t} + \boldsymbol{V}_{\mathrm{f}} \cdot \nabla\right)\boldsymbol{B}_0\right] \times \mathrm{d}\boldsymbol{l}_0\mathrm{d}t \tag{2.32}$$

when second order terms are neglected and we remember that $\boldsymbol{B}_0 \times \mathrm{d}\boldsymbol{l}_0 = 0$ by definition. Since $\mathrm{d}\boldsymbol{l}_0 = \mathrm{d}l_0\boldsymbol{B}_0/B_0$ equation (2.32) can also be written as

$$\begin{aligned} \boldsymbol{B}\times\mathrm{d}\boldsymbol{l} &= \mathrm{d}l_0\mathrm{d}t\, \hat{\boldsymbol{B}}_0 \times \left[(\boldsymbol{B}_0 \cdot \nabla)\boldsymbol{V}_{\mathrm{f}} - \left(\boldsymbol{V}_{\mathrm{f}} \cdot \nabla + \frac{\partial}{\partial t}\right)\boldsymbol{B}_0\right] \\ &= \mathrm{d}l_0\mathrm{d}t\, \hat{\boldsymbol{B}}_0 \times \left[\mathrm{curl}\,(\boldsymbol{V}_{\mathrm{f}} \times \boldsymbol{B}_0) - \boldsymbol{V}_{\mathrm{f}}\mathrm{div}\ \boldsymbol{B}_0 + \boldsymbol{B}_0\mathrm{div}\ \boldsymbol{V}_{\mathrm{f}} + \mathrm{curl}\ \boldsymbol{E}_0\right] \end{aligned} \tag{2.33}$$

when (2.1) and some well-known vector identities are used. The condition for line-preservation is that $\boldsymbol{B}$ and $\mathrm{d}\boldsymbol{l}$ always should be parallel, i.e., when subscript ($_0$) is dropped:

$$\boldsymbol{B} \times \mathrm{curl}\,(\boldsymbol{E} + \boldsymbol{V}_{\mathrm{f}} \times \boldsymbol{B}) = 0. \tag{2.34}$$

Comparison between equations (2.29) and (2.34) shows that every flux-

preserving field $\boldsymbol{V}_\mathrm{f}$ is also line-preserving, but the inverse statement is not necessarily true.

To connect the present results with the concept of moving field lines we now state that there exists a family of lines L_v moving with the velocity $\boldsymbol{V}_\mathrm{f}$ and having the following properties (cf. NEWCOMB [1958]):

(a) Through every point in space passes not more than one of the lines L_v.

(b) The lines L_v remain tangent to $\boldsymbol{B}$ during their motion.

(c) The density of lines L_v is proportional to the intensity of the magnetic field.

(d) The magnitude of the electromotive force induced around a closed curve C moving in an arbitrary manner with a velocity $\boldsymbol{V}_F$ is equal to the total number of lines L_v cut by the circuit per unit time. The sign of the same force is such that it tends to induce a current which preserves the magnetic flux enclosed by C.

First we must show that the statements (a) to (d) imply that $\boldsymbol{V}_\mathrm{f}$ is flux-preserving. Thus, assume that statements (a) to (d) are true and choose $\boldsymbol{V}_F$, equal to $\boldsymbol{V}_\mathrm{f}$. Then C and L_v move with the same velocity and the number of lines L_v inside C must be constant and equal to the magnetic flux through C according to (b) and (c).

Inversely, start with the assumption that $\boldsymbol{V}_\mathrm{f}$ is flux-preserving and consider the family of lines L_v, which coincide at the time t_0 with the magnetic lines of force. Since $\boldsymbol{V}_\mathrm{f}$ is flux-preserving it is also line-preserving as shown by (2.34), and conditions (a), (b) and (c) are satisfied. Finally, the electric field measured in a coordinate system following C is $\boldsymbol{E} + \boldsymbol{V}_F \times \boldsymbol{B}$ if it is $\boldsymbol{E}$ in the laboratory system. The electromotive force in the moving circuit C is therefore

$$\begin{aligned}\phi_\mathrm{C} &= \oint_\mathrm{C} (\boldsymbol{E} + \boldsymbol{V}_F \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{l} \\ &= \oint_\mathrm{C} (\boldsymbol{E} + \boldsymbol{V}_\mathrm{f} \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{l} - \oint_\mathrm{C} \boldsymbol{B} \cdot [(\boldsymbol{V}_F - \boldsymbol{V}_\mathrm{f}) \times \mathrm{d}\boldsymbol{l}], \qquad (2.35)\end{aligned}$$

where the first integral of the right hand member vanishes since $\boldsymbol{V}_\mathrm{f}$ is flux-preserving. The second integral of the same member has just the form stated by (d) and (c).

Consequently, we have shown that statements (a) to (d) are equivalent to the statement that $\boldsymbol{V}_\mathrm{f}$ is flux-preserving. We can then picture the magnetic field lines as moving with the velocity $\boldsymbol{V}_\mathrm{f}$.

Observe that the present analysis breaks down at a zero point of the magnetic field where condition (a) does not apply. The field line motion in presence of such points has been analysed by NEWCOMB [1955] in a number of special cases. In an approach which is more rigorous than that presented in this paragraph, the concept of a density of field lines has to be based on measure theory.

The present results will be applied later to the velocity fields discussed in Ch. 4, § 1.3 and in Ch. 6, § 1.

2. The Equation of Motion

2.1. CONSERVATION LAWS

Consider a particle of mass m and charge q with position given by $\rho(t) = (x(t), y(t), z(t))$. The particle is assumed to move with the velocity $\boldsymbol{w} = \mathrm{d}\rho/\mathrm{d}t$ in a force field $\boldsymbol{F}$ arising from the electric field $\boldsymbol{E}$ and from the gravitation potential ϕ_g.

Balance of momentum is expressed by

$$m \frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t} = \boldsymbol{F} + q\boldsymbol{w} \times \boldsymbol{B}, \tag{2.36}$$

where

$$\boldsymbol{F} = -\nabla(q\phi + m\phi_g) - q\frac{\partial \boldsymbol{A}}{\partial t} = q\boldsymbol{E} - m\nabla\phi_g. \tag{2.37}$$

Scalar multiplication of (2.36) by $\boldsymbol{w}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}mw^2) = \boldsymbol{F}\cdot\boldsymbol{w} \tag{2.38}$$

which expresses conservation of energy. Since (2.38) is deduced from (2.36) it does not give any new information beyond that obtained from conservation of momentum. In the macroscopic theory of Chapter 5 the situation becomes different because there the total velocity $\boldsymbol{w}$ is divided into one thermal part and one part representing a mass motion. These two parts then have to be connected by an additional condition which represents the conservation of energy.

The force $q\boldsymbol{w} \times \boldsymbol{B}$ does not perform any work on the particle. In a purely magnetostatic field where $\boldsymbol{F} = 0$ the energy $\frac{1}{2}mw^2$ therefore remains constant.

According to (2.37) and (2.38) the change in energy per unit time becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}mw^2\right) = -\boldsymbol{w}\cdot\nabla\,(q\phi + m\phi_g) - q\,\frac{\partial \boldsymbol{A}}{\partial t}\cdot\boldsymbol{w}. \tag{2.39}$$

The first term of the right hand member of (2.39) is due to the change in potential energy of the particle when it moves across the equipotential surfaces given by $q\phi + m\phi_g$ and the second term represents a "betatron" acceleration by electromagnetic induction.

For a treatment of the equation of motion in a rotating coordinate system reference is made to Ch. 7, § 2.2.

*2.2. SCALING LAWS

The equation of motion can be written in dimensionless form by means of the transformations

$$x_k = L_c\cdot x_k',\quad t = t_c\cdot t',\quad \boldsymbol{A} = A_c\cdot\boldsymbol{A}',\quad \phi = \phi_c\cdot\phi',\quad \phi_g = \phi_{gc}\cdot\phi_g', \tag{2.40}$$

where x_k indicate space coordinates and primed quantities are dimensionless variables. The coefficients with subscript ($_c$) are constants with the same dimension as the unprimed quantities and can be considered to represent characteristic values of the latter. The particular choice of these values is immaterial. The equation of motion can be written as:

$$m\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t} = -\nabla(q\phi + m\phi_g) - q\,\frac{\partial \boldsymbol{A}}{\partial t} + q\boldsymbol{w}\times\operatorname{curl}\boldsymbol{A}. \tag{2.41}$$

By means of the substitutions (2.40) it is brought into the form

$$k_1\frac{\mathrm{d}^2\boldsymbol{\rho}'}{\mathrm{d}t'^2} = -k_2\nabla'\phi' - k_3\nabla'\phi_g' - \frac{\partial \boldsymbol{A}'}{\partial t'} + \frac{\mathrm{d}\boldsymbol{\rho}'}{\mathrm{d}t'}\times\operatorname{curl}'\boldsymbol{A}' \tag{2.42}$$

with the dimensionless parameters

$$k_1 = mL_c/qA_ct_c\,,\quad k_2 = t_c\phi_c/A_cL_c\,,\quad k_3 = mt_c\phi_{gc}/qA_cL_c\,. \tag{2.43}$$

Consider a given solution of (2.42) for the primed quantities with its corresponding boundary conditions. By keeping all three parameters k_1, k_2 and k_3 constant and varying the characteristic quantities with subscript ($_c$) it is then possible to generate a whole set of solutions (2.40) of the equation of motion (2.41). The result of this procedure is defined as a set of *similar configurations*. For a member of such a set a field quantity at a certain point in space and time becomes proportional to the same quantity of any other

member of the set at corresponding points in space and time, as is immediately obvious from expressions (2.40). Also the relative magnitude of the terms in the equation of motion (2.41) is constant at corresponding points within a set of similar configurations.

To take a specific example, a change in length and time scale within such a set results in a change by a certain factor in the characteristic velocity L_c/t_c. This requires a change by the same factor in the characteristic potential A_c and by the square of the same factor in the potentials ϕ_c and ϕ_{gc} as indicated by equations (2.43). Thus, the product of k_1 and k_2 for similar configurations results in $\phi_c \propto A_c^2$, which is equivalent to a condition earlier derived by BLOCK [1956].

Besides being regarded as transformations to dimensionless variables relations (2.42) and (2.43) can also be used to estimate the order of magnitude of the different terms in the equation of motion (2.41). This requires a closer specification of the characteristic quantities as will be demonstrated later in Ch. 3, § 1.2. Observe that primed terms are not necessarily of order unity.

In the scaling laws of this volume we will only take those effects into account which are represented in the equation of motion (2.41). There may exist other effects such as collision phenomena, which have not been included here but which play an important rôle in a plasma under experimental conditions. For further discussions on this problem reference is made to ALFVÉN [1950].

3. Hamiltonian Formalism

Instead of working directly with the equation of motion an alternative structure of the theory can be used which is known as the Hamiltonian formulation. This does not add anything new to the physics involved but provides a more powerful theoretical method.

3.1. CANONICAL EQUATIONS

For a particle moving with velocity $\boldsymbol{w}$ the vector identity

$$\nabla (\boldsymbol{w} \cdot \boldsymbol{A}) = (\boldsymbol{w} \cdot \nabla) \boldsymbol{A} + \boldsymbol{w} \times \operatorname{curl} \boldsymbol{A} \tag{2.44}$$

holds. The total time derivative of $\boldsymbol{A}$ observed in a coordinate system following the particle becomes

$$\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} \equiv \left(\frac{\partial}{\partial t} + \boldsymbol{w} \cdot \nabla\right) \boldsymbol{A} \ . \tag{2.45}$$

By means of expressions (2.44) and (2.45) the equation of motion (2.41) can be written in the form

$$m \frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t} = -\nabla(m\phi_g + q\phi - q\boldsymbol{w} \cdot \boldsymbol{A}) - q \frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} . \tag{2.46}$$

If the *Lagrangian*

$$L = \tfrac{1}{2}mw^2 - m\phi_g - q\phi + q\boldsymbol{w} \cdot \boldsymbol{A} \tag{2.47}$$

is introduced equation (2.46) is easily seen to correspond to the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_k}\right) - \frac{\partial L}{\partial x_k} = 0, \tag{2.48}$$

where $\dot{x}_k = w_k$ are time derivatives of the coordinates x_k. In absence of a magnetic field the Lagrangian of equation (2.47) is equal to the difference between the kinetic and potential energies of the particle. In presence of such a field, however, it is not clear whether the interaction represented by $q\boldsymbol{w} \cdot \boldsymbol{A}$ should be considered as a kinetic energy, because of its dependence on $\boldsymbol{w}$, or as a potential energy, since it depends on the external field $\boldsymbol{A}$. As a matter of fact, we shall see that such a classification is not necessary for the development of the theory.

There are often certain constraints which limit the motions of a physical system. When the conditions of constraint can be expressed by equations connecting the coordinates of the particle it is useful to introduce a set $q_k = q_k(t)$ of *generalized coordinates* which represent the degrees of freedom of the system. The Lagrangian then still has the form (2.47) and Lagrange's equations can now be shown to become:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0. \tag{2.49}$$

Equations (2.49) are identical with the differential equations resulting from the variation problem

$$\delta \int_{t_1}^{t_2} L \, \mathrm{d}t = 0. \tag{2.50}$$

expressing *Hamilton's principle*, where the times t_1 and t_2 are fixed.

The Lagrangian is a function of the coordinates q_k, the "velocities" $\dot{q}_k$ and of time t. We further define a *generalized momentum*

$$p_k = \frac{\partial}{\partial \dot{q}_k} L(q_j, \dot{q}_j, t) \tag{2.51}$$

and a *Hamiltonian*

$$H = \sum_k p_k \dot{q}_k - L(q_k, \dot{q}_k, t). \tag{2.52}$$

The definition (2.51) implies that the generalized momentum is a function of q_j, $\dot{q}_j$ and t. Inversely, this definition permits the elimination of the "velocities" $\dot{q}_j$ so that the system can now be described in terms of q_k, p_k and t which have to be considered as independent variables in the canonical equations. The differential of H becomes

$$\mathrm{d}H = \sum_k \left(p_k - \frac{\partial L}{\partial \dot{q}_k}\right)\mathrm{d}\dot{q}_k + \sum_k \left(\dot{q}_k \mathrm{d}p_k - \frac{\partial L}{\partial q_k}\mathrm{d}q_k\right) - \frac{\partial L}{\partial t}\mathrm{d}t. \tag{2.53}$$

In this equation the first term of the right hand member vanishes according to the definition (2.51). Thus, it is possible to express $\mathrm{d}H$ by the changes in p_k, q_k and t alone. From the expression of $\partial L/\partial q_k$ given by (2.49) and from equations (2.51) and (2.53) we now obtain

$$\mathrm{d}H = \sum_k (\dot{q}_k \mathrm{d}p_k - \dot{p}_k \mathrm{d}q_k) - \frac{\partial L}{\partial t}\mathrm{d}t \equiv \sum_k \left(\frac{\partial H}{\partial p_k}\mathrm{d}p_k + \frac{\partial H}{\partial q_k}\mathrm{d}q_k\right) + \frac{\partial H}{\partial t}\mathrm{d}t. \tag{2.54}$$

Consequently,

$$\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t} \tag{2.55}$$

and

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \tag{2.56}$$

$$\dot{p}_k = -\frac{\partial H}{\partial q_k}, \tag{2.57}$$

where the two last relations are *Hamilton's canonical equations.*

It may sometimes be useful to make a transformation from one set of coordinates to another. In the present Hamiltonian formulation the generalized momenta of (2.51) are also independent variables on the same level as the generalized coordinates. A general transformation will therefore include a simultaneous change of both the coordinates and the momenta (q_k, p_k) to a new set $q'_k = q'_k(q_k, p_k, t)$ and $p'_k = p'_k(q_k, p_k, t)$. Of particular interest here are such new variables which yield a set of equations of the canonical form (2.56) and (2.57) with a new Hamiltonian $H' = H'(q'_k, p'_k, t)$ and a

corresponding Lagrangian L' connected with H' by an equation analogous to (2.52). Transformations with this property are said to be *canonical.*

If q'_k and p'_k are to be canonical coordinates they must satisfy Hamilton's principle (2.50) in the same way as the original coordinates q_k and p_k. The corresponding Lagrangians L' and L then differ at most by a total time derivative of an arbitrary function G. The latter must be a function of both the new and the original variables to affect the transformation. Since q'_k and p'_k are functions of q_k and p_k we can always express G as a function of two of the four variables q_k, p'_k, q_k and p_k.

There are four alternatives of which one possible choice becomes $G = G_1(\dot{q}_k, \dot{q}'_k, t)$. We then expand the total time derivative $\mathrm{d}G_1/\mathrm{d}t$ and put it equal to the difference $L - L'$ in the form given by (2.52). The condition that the coefficients of $\dot{q}_k$ and $\dot{q}'_k$ in the obtained expression should vanish separately leads, among other things, to $\partial G_1/\partial q'_k = -p'_k$. In later discussions we shall become more interested in a form of G which can be written as $G_2(q_k, p'_k, t) = G_1(q_k, q'_k, t) + \sum p'_k q'_k$. With this relation substituted into $\mathrm{d}G_1/\mathrm{d}t = L - L'$ and by requiring that the coefficients of $\dot{q}_k$ and $\dot{p}'_k$ should vanish, the corresponding transformation equations become:

$$p_k = \frac{\partial G_2}{\partial q_k}, \quad q'_k = \frac{\partial G_2}{\partial p_k'}, \quad H' = H + \frac{\partial G_2}{\partial t}. \tag{2.58}$$

Consequently, as soon as G_2 is given the transformation becomes completely specified. Therefore G_2 is called a *generating function* of the canonical transformation. The three additional forms, $G_1(q_k, q'_k, t)$, $G_3(p_k, p'_k, t)$ and $G_4(q'_k, p_k, t)$, in which the generating function G can be chosen, give rise to conditions similar to equations (2.58).

3.2. CONSTANTS OF THE MOTION

The total time derivative of an arbitrary function $\chi = \chi(q_k, p_k, t)$ is

$$\begin{aligned}\frac{\mathrm{d}\chi}{\mathrm{d}t} &= \frac{\partial \chi}{\partial t} + \sum_k \left(\frac{\partial \chi}{\partial q_k} \dot{q}_k + \frac{\partial \chi}{\partial p_k} \dot{p}_k \right) \\ &= \frac{\partial \chi}{\partial t} + \sum_k \left(\frac{\partial \chi}{\partial q_k} \cdot \frac{\partial H}{\partial p_k} - \frac{\partial \chi}{\partial p_k} \cdot \frac{\partial H}{\partial q_k} \right) \equiv \frac{\partial \chi}{\partial t} + \{\chi, H\},\end{aligned} \tag{2.59}$$

where use has been made of the canonical equations (2.56) and (2.57). The expression $\{\chi, H\}$ defined by the last two members of (2.59) is denoted as the Poisson bracket of χ and H. If a quantity χ does not depend explicitly

on time and if its Poisson bracket with H vanishes, the total time derivative of χ is zero. Then, χ is a constant of the motion.

An important special case is that where the Hamiltonian does not depend explicitly on time. According to (2.59) it is then a constant of the motion, and we always have $H(q_k, p_k) = H_0 = \text{const.}$, whatever canonical variables q_k, p_k are used to describe the system. If a canonical transformation is made to the new variables q'_k, p'_k, the new Hamiltonian therefore becomes $H'(q'_k, p'_k) = H(q_k, p_k) = H_0$.

In the situation where $\partial H/\partial t = 0$ we shall now try to find a canonical transformation which leads to the result that all the momenta p'_k in the new representation become constants of the motion. Thus, we require $\dot{p}'_k = 0$ and $p'_k \equiv \alpha_k = \text{const.}$ The generating function used in § 3.1 should then have the form $G = W(q_1, q_2, \ldots, \alpha_1, \alpha_2, \ldots)$ and the corresponding transformation (2.58) leads to

$$p_k = \frac{\partial W}{\partial q_k}, \quad q'_k = \frac{\partial W}{\partial \alpha_k}, \quad H\left(q_1, q_2, \cdots, \frac{\partial W}{\partial q_1}, \frac{\partial W}{\partial q_2}, \cdots\right) = H_0. \tag{2.60}$$

The last of expressions (2.60) is the *Hamilton-Jacobi equation* for Hamilton's characteristic function $W(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n)$, where n is the number of degrees of freedom.

The Hamilton-Jacobi equation is a partial differential equation for W with a number of constants of integration equal to the number n of degrees of freedom. Only the derivatives of W appear in the equation. If W is a solution, then $W + \text{const.}$ must also be so. The new momenta α_k have not yet been specified, except that we know that they must be constants. We are therefore at liberty to take the n constants of integration to be the momenta α_k. Thus, the complete solution of the Hamilton-Jacobi equation (2.60) contains $n - 1$ non-trivial constants of integration, $\alpha_2, \ldots, \alpha_n$, and one quantity α_1 which is merely an additive constant. When evaluated at the initial time t_0 the first of relations (2.60) serves to determine these n independent constants of integration with the initial values of q_k and p_k. The constant H_0 is determined by these values and is simply a combination of the n constants of integration. Without loss of generality we can choose $p'_1 \equiv \alpha_1 = H_0$.

The solution W is specified by the initial conditions. We can further determine the momenta p_k in terms of q_k from the first of expressions (2.60). Finally, the canonical equation (2.56) for q'_k is $\dot{q}'_k = \partial H'/\partial \alpha_k = \partial H_0/\partial \alpha_k$, which vanishes when $k \neq 1$ and is equal to unity when $k = 1$. In combi-

nation with the second of expressions (2.60) this yields relations from which the coordinates q_k can be solved as functions of time. The solution of the problem is then completed.

Before closing this paragraph we shall devote some space to the question how the Hamiltonian is related to the total energy of the system. In the simple situation where a rectangular system can be chosen as generalized coordinates the corresponding momenta become according to equations (2.47) and (2.51)

$$p_k = mw_k + qA_k. \tag{2.61}$$

From (2.52) the Hamiltonian then reduces to

$$H = \tfrac{1}{2}mw^2 + q\phi + m\phi_g \tag{2.62}$$

which is equal to the total energy of the particle. In particular, if $\partial H/\partial t = 0$ we have here a situation where the Hamiltonian is both a constant of the motion and is equal to the total energy.

However, in a general situation the Hamiltonian does not necessarily become equal to the total energy when it is a constant of the motion and *vice versa*. The reason for this is that the Lagrangian contains only the work of the external forces and not that of the forces of constraint. When the latter are time-dependent they will perform a work on the particle. One example of this is given by a bead sliding on a moving wire, where the position and motion of the bead is described by a generalized coordinate measuring the arc length along the wire. As another example we may take a charged particle moving in a time-dependent magnetic field and where the position of the particle is described by the "magnetic field coordinates" s, α, β of § 1.2 in the present chapter. The moving field lines with their associated coordinates will then become moving constraints.

3.3. PERIODIC AND NEARLY PERIODIC MOTION

Systems in which the motion is periodic are of special importance in many branches of physics. In a system with one degree of freedom phase space consists of a plane defined by the coordinates q_1 and p_1. There are two types of periodic motion, namely libration and rotation. For the former the orbit in phase space is a closed curve and for the latter q_1 increases indefinitely with time whereas p_1 is a periodic function of q_1.

For a system with more than one degree of freedom we shall here restrict ourselves to such situations where the projections of its orbit on each plane (q_k, p_k) is periodic in the sense just defined for one degree of freedom. We shall treat such systems for which the periodic motions in each (q_k, p_k) plane

are *independent* of each other, and where the Hamilton-Jacobi equation (2.60) for W is separable in at least one set of canonical variables. Thus, Hamilton's characteristic function $W = W(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n)$ should be separable in a sum of n terms of the form $W_k(q_k, \alpha_1, \ldots, \alpha_n)$. From the first of expressions (2.60) we then see that p_k is a function of q_k and $\alpha_1, \ldots, \alpha_n$ only. Since all the momenta p_k are periodic functions of the corresponding coordinates q_k we can then form a set of n quantities defined by

$$J_k = \oint p_k \mathrm{d}q_k = \oint \frac{\partial W_k}{\partial q_k} \mathrm{d}q_k = J_k(\alpha_1, \cdots, \alpha_n), \tag{2.63}$$

where integration should be performed over a complete period of libration or rotation in each (q_k, p_k) plane. The n *action variables* J_k defined by (2.63) are functions only of the n constants $\alpha_1, \ldots, \alpha_n$. Since p_k is a periodic function of q_k the same surface area J_k is retraced in the (q_k, p_k) plane during every period.

There often occur slow changes in the parameters of an oscillating physical system and the corresponding motion can be designated as "*nearly periodic*". Examples of this are given by a harmonic oscillator with slowly changing elasticity and by a charged particle which gyrates in a magnetic field which varies slowly in space and time. The action variables are then no longer exact constants. However, if the changes in the parameters are slow enough, one would expect the deviations of the action variables from constancy to become small. A nearer examination of this question leads to the problem of the so called *adiabatic invariance* of the action variables. This problem was first discussed by BURGERS [1917] and KRUTKOW [1918] in earlier attempts to describe the mechanics of atoms, and has later attracted new interest in the field of plasma physics. As stated by BORN [1925] the problem can be formulated as follows:

(i) Consider a mechanical system which in an undisturbed state performs a periodic motion as earlier defined in this paragraph. The system is then associated with a number of action variables J_k which are constants in the undisturbed state.

(ii) In the equations of motion of the same system we further introduce an additional parameter which is a function of time.

(iii) We now define adiabatic changes of the system as such changes due to the additional parameter which are *not related* to the periods of the system, and which proceed at an *infinitely slow* rate compared to these periods. By

the first part of this definition we exclude resonance phenomena between the periods of the system and the changes in the additional parameter.

Proofs for the adiabatic invariance of the action variables have earlier been given by Burgers, Krutkow and Born for systems with many degrees of freedom. We shall not repeat their analysis here, but shall instead develop an expression for the action integral of a nearly periodic system by a method introduced by KRUSKAL [1957, 1958] and discussed by BRINKMAN [1959, 1960]. For this purpose the canonical variables are written in the form

$$q_k = q_k[\Theta(t), t], \qquad p_k = p_k[\Theta(t), t], \tag{2.64}$$

where Θ is a periodic function of time and the explicit dependence of time represents changes in q_k and p_k which are not in resonance with the periods of the system. With the definition (2.64) the canonical equations (2.56) and (2.57) become

$$\frac{\partial H}{\partial p_k} = \frac{\partial q_k}{\partial \Theta}\dot{\Theta} + \frac{\partial q_k}{\partial t}, \tag{2.65}$$

$$-\frac{\partial H}{\partial q_k} = \frac{\partial p_k}{\partial \Theta}\dot{\Theta} + \frac{\partial p_k}{\partial t}, \tag{2.66}$$

where the partial derivatives of the right hand members now refer to the special notation of equations (2.64). The Hamiltonian has the form

$$H = H\,[q_k(\Theta, t), p_k(\Theta, t), t] \tag{2.67}$$

from which follows that

$$\frac{\partial H}{\partial \Theta} = \sum_{k=1}^{n} \left(\frac{\partial H}{\partial q_k} \cdot \frac{\partial q_k}{\partial \Theta} + \frac{\partial H}{\partial p_k} \cdot \frac{\partial p_k}{\partial \Theta} \right). \tag{2.68}$$

Combination of equations (2.65), (2.66) and (2.68) yields

$$\begin{aligned} \frac{\partial H}{\partial \Theta} &= \sum_{k=1}^{n} \left(\frac{\partial q_k}{\partial t} \cdot \frac{\partial p_k}{\partial \Theta} - \frac{\partial p_k}{\partial t} \cdot \frac{\partial q_k}{\partial \Theta} \right) \\ &= \sum_{k=1}^{n} \left[\frac{\partial}{\partial \Theta} \left(p_k \frac{\partial q_k}{\partial t} \right) - \frac{\partial}{\partial t} \left(p_k \frac{\partial q_k}{\partial \Theta} \right) \right]. \end{aligned} \tag{2.69}$$

For a moment we consider Θ and t as if they were independent variables. In the expressions of the right and left hand members of (2.69) we then keep t constant in all places where it occurs explicitly and integrate over a period with respect to Θ. Since both H and $p_k\, \partial q_k/\partial t$ return to the same values

after one period under these restrictions, there are no contributions from these quantities to the integral. Thus,

$$\sum_{k=1}^{n} \oint \left[\frac{\partial}{\partial t}\left(p_k \frac{\partial q_k}{\partial \Theta}\right) \mathrm{d}\Theta\right]_t = 0, \tag{2.70}$$

where the subscript indicates that time should be kept constant in all places where it occurs explicitly in the sense of equation (2.64). Since Θ and t are treated as independent variables the order of integration and derivation can be changed in equation (2.70) and

$$J^* = \sum_{k=1}^{n} J_k^* = \text{const.}, \quad J_k^* = \oint \left[p_k \frac{\partial q_k}{\partial \Theta}\right]_t \mathrm{d}\Theta, \tag{2.71}$$

i.e., the action integral J^* becomes constant with respect to the explicit dependence of time. The result is purely formal and holds only as long as one can introduce the variable Θ in the way specified here. It has a physical meaning only when the motion of the particle can be split into a rapidly fluctuating periodic part and a slowly varying aperiodic one. The integral (2.71) then implies that an average is taken over the fluctuation, whereas the slowly varying part is kept constant.

For a system of one degree of freedom we have $J^* = J_1^*$. When the explicit time dependence in relations (2.64) becomes infinitely slow, the difference between J_1^* and the action variable J_1 of (2.63) tends to zero.

For a system with many degrees of freedom it is not obvious from (2.71) that each of the terms J_k^* should become a constant. However, there are cases where the motions in each (q_k, p_k) plane can be treated as independent events and the Hamiltonian (2.67) separates into a sum of corresponding terms. Each degree of freedom then behaves as a one-dimensional system and the analysis of equations (2.68) – (2.71) holds for every separate value of k. Then, J_k^* becomes constant and approaches J_k in the limit of an infinitely slow explicit time dependence.

4. Special Solutions of the Equation of Motion

An exact solution of the equations of motion can only be found in a few special cases. A number of examples will be given in the present paragraph which partly serve as illustrations to the contents of subsequent chapters.

4.1. HOMOGENEOUS MAGNETOSTATIC AND ELECTRIC FIELDS

Consider a particle moving in a homogeneous magnetostatic field $\boldsymbol{B}$ and a time-dependent homogeneous electric field $\boldsymbol{E}$. Induction effects associated with the time variation of $\boldsymbol{E}$ are neglected. Introduce a rectangular coordinate system as shown in Fig. 2.4.a with $\boldsymbol{B} = (0, 0, B)$ and with the electric field $\boldsymbol{E} = (E_x(t), 0, E_z(t))$ in the xz plane. The equation of motion (2.36) then results in

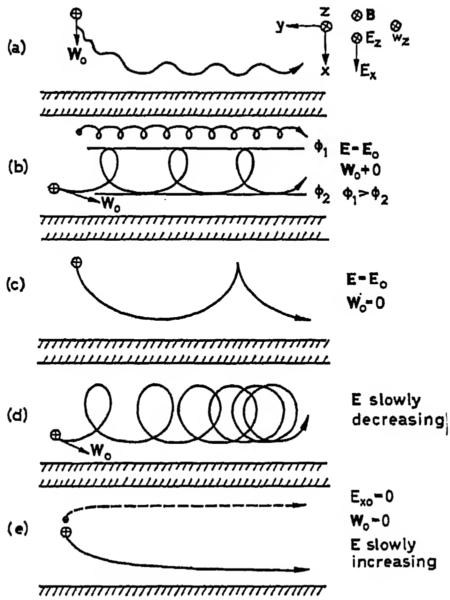


Fig. 2. 4. Orbits of charged particle in a homogeneous magnetostatic field $\boldsymbol{B}$ and a homogeneous, time-dependent electric field $\boldsymbol{E}$. a. Definition of coordinates. b. Constant electric field $\boldsymbol{E}_0$. The electric drift is the same for particles of both signs. c. Constant electric field and zero initial velocity $\boldsymbol{W}_0$. d. Electric field slowly decreasing. The rate of change of the electric field drift is shown on a strongly exaggerated scale. e. Electric field slowly increasing from zero. No initial velocity. Displacements in x direction strongly exaggerated.

$$\frac{\mathrm{d}w_x}{\mathrm{d}t} = \omega_g w_y + \omega_g E_x/B, \tag{2.72}$$

$$\frac{\mathrm{d}w_y}{\mathrm{d}t} = -\omega_g w_x, \tag{2.73}$$

$$\frac{\mathrm{d}w_z}{\mathrm{d}t} = qE_z/m, \tag{2.74}$$

where $\omega_g = qB/m$ is the frequency of gyration in the magnetic field, as will be seen later. The motion in the z direction, i.e. along the magnetic field, is a non-uniform acceleration according to (2.74) which separates from the motion in the xy plane and need not be considered further.

For the transverse motion substitution of w_x from (2.73) into (2.72) results in

$$\frac{d^2 w_y}{dt^2} + \omega_g^2 w_y = -\omega_g^2 E_x/B. \tag{2.75}$$

The initial conditions are given at $t = 0$ where $w_y(0) = w_{y0}$, $w_x(0) = w_{x0}$ and $E_x(0) = E_{x0}$. The general solution of (2.75) consists of the general solution of the corresponding homogeneous equation together with a particular solution where the right hand member is included. With the applied initial conditions it is easily seen that the result becomes

$$\begin{aligned} w_y = {} & w_{y0} \cos \omega_g t - w_{x0} \sin \omega_g t \\ & + (\omega_g/B) \cos \omega_g t \int_0^t E_x \sin \omega_g t dt - (\omega_g/B) \sin \omega_g t \int_0^t E_x \cos \omega_g t dt, \end{aligned} \tag{2.76}$$

$$\begin{aligned} w_x = {} & w_{x0} \cos \omega_g t + w_{y0} \sin \omega_g t \\ & + (\omega_g/B) \sin \omega_g t \int_0^t E_x \sin \omega_g t dt + (\omega_g/B) \cos \omega_g t \int_0^t E_x \cos \omega_g t dt. \end{aligned} \tag{2.77}$$

We further integrate the expressions for w_y and w_x partially once and twice, respectively (LEHNERT [1962b]):

$$\begin{aligned} w_y = {} & w_{y0} \cos \omega_g t - w_{x0} \sin \omega_g t + (E_{x0}/B) \cos \omega_g t - E_x/B \\ & + (1/B) \cos \omega_g t \int_0^t \frac{dE_x}{dt} \cos \omega_g t dt + (1/B) \sin \omega_g t \int_0^t \frac{dE_x}{dt} \sin \omega_g t dt \end{aligned} \tag{2.78}$$

and

$$\begin{aligned} w_x = {} & w_{x0} \cos \omega_g t + w_{y0} \sin \omega_g t + (E_{x0}/B) \sin \omega_g t \\ & + (1/\omega_g B) \frac{dE_x}{dt} - (1/\omega_g B) \left(\frac{dE_x}{dt}\right)_0 \cos \omega_g t \\ & - (1/\omega_g B) \cos \omega_g t \int_0^t \frac{d^2 E_x}{dt^2} \cos \omega_g t dt - (1/\omega_g B) \sin \omega_g t \int_0^t \frac{d^2 E_x}{dt^2} \sin \omega_g t dt, \end{aligned} \tag{2.79}$$

where subscript $(_0)$ refers to initial values.

Qualitative discussions on the particle orbits in a time-dependent transverse electric field have earlier been made by ANDERSON *et al.* [1958, 1959]. Here we shall use equations (2.78) and (2.79) to treat the following special cases:

(i) $\boldsymbol{E} = 0$

The projection of the particle orbit in the xy plane is a circle along which the particle moves with the constant velocity $W_0 = (w_{x0}^2 + w_{y0}^2)^{\frac{1}{2}}$. The velocity at the starting point is $\boldsymbol{W}_0$. Integration of equations (2.78) and (2.79) shows that the position of the particle can be described by a vector

$$\boldsymbol{a} = \frac{1}{\omega_g}\hat{\boldsymbol{B}} \times \boldsymbol{W}_0 \cos \omega_g t + \frac{1}{\omega_g}\boldsymbol{W}_0 \sin \omega_g t, \quad \boldsymbol{W}_0 = (w_{x0}, w_{y0}, 0), \tag{2.80}$$

extended from the *centre of gyration* (or *guiding centre*), O. The unit vector along $\boldsymbol{B}$ is denoted by $\hat{\boldsymbol{B}} = \boldsymbol{B}/B$ and the sign of ω_g is given by the sign of the particle charge q. We define the modulus of $\boldsymbol{a}$ as the *radius of gyration* (or *Larmor radius*),

$$a = W_0/\,|\,\omega_g\,| = mW_0/\,|\,q\,|\,B. \tag{2.81}$$

For a given velocity W_0 it becomes inversely proportional to the field strength B. The changes of the vector $\boldsymbol{a}$ correspond to a *velocity of gyration*

$$\boldsymbol{W} = \mathrm{d}\boldsymbol{a}/\mathrm{d}t = \omega_g \boldsymbol{a} \times \hat{\boldsymbol{B}}. \tag{2.82}$$

The motion of the particle around the magnetic field lines can be represented by a circulating electric current and an *equivalent magnetic moment*

$$\boldsymbol{M} = -\tfrac{1}{2}m\boldsymbol{B}(W_0/B)^2. \tag{2.83}$$

Observe that, for both signs of the charge q, this moment is antiparallel with $\boldsymbol{B}$. Thus, the particle moves in a way such as to create an induced magnetic field which is opposite to the external field at the centre of gyration. One would therefore expect a gas of charged particles to have diamagnetic properties.

Since $E_z = 0$ the motion along $\boldsymbol{B}$ takes place at constant velocity and the total orbit becomes a helix with a constant pitch given by w_z/W as shown in Figure 2.5.

(ii) $\boldsymbol{E} = \boldsymbol{E}_0 = \text{constant}$

Only the first four terms of the right hand member of (2.78) remain as well as the first three terms of the right hand member of (2.79). By the substitution

$$w_y' = w_y + E_{x0}/B, \qquad w_{y0}' = w_{y0} + E_{x0}/B \tag{2.84}$$

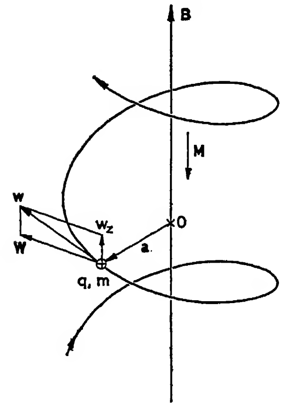


Fig. 2.5. Motion of charged particle in homogeneous magnetostatic field $\boldsymbol{B}$.

equations (2.78) and (2.79) are brought back to the form just discussed in (i). Therefore, the motion consists of a gyration with velocity $\boldsymbol{W}$ superimposed on a translation with velocity $\boldsymbol{E}_0 \times \boldsymbol{B}/B^2$. This is also immediately seen from the equation of motion (2.36) for which the transverse electric field vanishes if the particle motion is observed in a coordinate system moving with the velocity $\boldsymbol{E}_0 \times \boldsymbol{B}/B^2$.

The projection of the particle orbit in the xy plane is as shown in Figure 2.4b. When the longitudinal field E_z vanishes the particle drifts in the y direction in the space between two equipotentials given by $\phi = \phi_1$ and $\phi = \phi_2$. The drift motion is also understood from the fact that the particle increases its transverse velocity when it "falls" from ϕ_1 to ϕ_2. The instantaneous radius of gyration then becomes smaller at ϕ_1 then at ϕ_2. Observe that the electric drift is the same for particles of both signs. Especially when the initial velocities w_{x0} and w_{y0} are zero the orbit becomes a cycloid as given by Figure 2.4c.

(iii) Periodic electric field $\boldsymbol{E}$

For a transverse electric field

$$E_x = E_{x0} \cos \omega t \tag{2.85}$$

oscillating with the frequency ω equations (2.76) and (2.77) can immediately be integrated. After applying some well-known trigonometric expressions the result becomes

$$w_y = w_{y0} \cos \omega_g t - w_{x0} \sin \omega_g t \tag{2.86}$$
$$+ [\omega_g E_{x0}/B(\omega + \omega_g)] \sin [\tfrac{1}{2}(\omega + \omega_g)t] \cdot \sin [\tfrac{1}{2}(\omega - \omega_g)t]$$
$$- [\omega_g E_{x0}/B(\omega - \omega_g)] \sin [\tfrac{1}{2}(\omega + \omega_g)t] \cdot \sin [\tfrac{1}{2}(\omega - \omega_g)t]$$

and

$$w_x = w_{x0} \cos \omega_g t + w_{y0} \sin \omega_g t + (E_{x0}/2B) \sin \omega_g t \tag{2.87}$$
$$- [E_{x0}(\omega - \omega_g)/2B(\omega + \omega_g)] \sin [\tfrac{1}{2}(\omega + \omega_g)t] \cdot \cos [\tfrac{1}{2}(\omega - \omega_g)t]$$
$$+ [E_{x0}(\omega + \omega_g)/2B(\omega - \omega_g)] \cdot \cos [\tfrac{1}{2}(\omega + \omega_g)t] \cdot \sin [\tfrac{1}{2}(\omega - \omega_g)t].$$

Of special interest is the case of cyclotron resonance where $\omega = \omega_g$ and equations (2.86) and (2.87) reduce to

$$w_y = w_{y0} \cos \omega_g t - w_{x0} \sin \omega_g t - (E_{x0}/2B)\, \omega_g t \sin \omega_g t \tag{2.88}$$

and

$$w_x = w_{x0} \cos \omega_g t + w_{y0} \sin \omega_g t + (E_{x0}/2B)(\sin \omega_g t + \omega_g t \cos \omega_g t). \tag{2.89}$$

For a sufficiently long time the last terms of these expressions will dominate and the transverse kinetic energy approaches the value

$$\tfrac{1}{2}m(w_x^2 + w_y^2) \approx (qE_{x0}t)^2/8m. \tag{2.90}$$

When the field E_x is tuned to the gyro frequency of the particle the latter therefore gains energy indefinitely according to the present theory. In reality the energy increase will be limited by radiation and frictional losses. For further discussions on mechanisms which can be used to increase the energy of an ionized gas reference is made to Ch. 6, § 2.

(iv) Slowly changing electric field $\boldsymbol{E}$

When the transverse electric field changes slowly compared with the gyro frequency, i.e. when $|(\mathrm{d}E_x/\mathrm{d}t)/(\omega_g E_x)|$ is much less than unity, the integrals of equations (2.78) and (2.79) become very small and can be neglected. This is easily seen when $E_x(t)$ is expanded in a power series in t. If the subsitution (2.84) is introduced it is found that the motion in the present approximation will consist of a gyration with velocity $\boldsymbol{W}$ superimposed on a translation with the drift velocity $\boldsymbol{E} \times \boldsymbol{B}/B^2$ which varies slowly in time.

A sketch of the orbit in a slowly decreasing field E_x is given in Figure 2.4d, where the rate of change of the electric drift $\boldsymbol{E} \times \boldsymbol{B}/B^2$ is shown on a strongly exaggerated scale and the small displacements in the x direction have been left out.

Especially if the particle starts from rest in a field E_x which increases slowly and monotonically from the initial value $E_{x0} = 0$ the orbits for an ion and an electron become as demonstrated by Figure 2.4e. Here the displacements in the x direction have been strongly exaggerated. Observe that ions and electrons are displaced oppositely in the x direction. In an ionized gas this gives rise to an electric polarization.

4.2. MAGNETOSTATIC FIELD WITH AXIAL SYMMETRY

We introduce cylindrical coordinates (r, φ, z) for a charged particle moving in the static, axially symmetric magnetic field

$$\boldsymbol{B} = \operatorname{curl} \boldsymbol{A} = \left(-\frac{\partial A_\varphi}{\partial z}, \ \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \ \frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi) \right). \tag{2.91}$$

Denote the particle velocity by $\boldsymbol{w} = (\dot{r}, r\dot{\varphi}, \dot{z})$, where a dot indicates total time derivative.

In the absence of electric and gravitation fields conservation of energy (2.38) yields

$$w_r^2 + w_\varphi^2 + w_z^2 = w_0^2. \tag{2.92}$$

The φ component of the equation of motion (2.36) leads to the relation

$$(m/q)\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\varphi}) = -r\frac{\partial A_\varphi}{\partial z}\dot{z} - \frac{\partial}{\partial r}(rA_\varphi)\dot{r} = -\frac{\mathrm{d}}{\mathrm{d}t}(rA_\varphi) \tag{2.93}$$

since A_φ does not depend explicitly on time. Integration of (2.93) yields

$$p_\varphi \equiv mrw_\varphi + qrA_\varphi = mr_0w_{\varphi 0} + qr_0A_{\varphi 0} = p_{\varphi 0}, \tag{294}$$

where subscript ($_0$) refers to values at the starting point of the particle. Relations (2.92) and (2.94) can also be obtained from the Hamiltonian theory of § 3.2. Equation (2.94) expresses conservation of the generalized angular momentum p_φ.

We apply these results to the following special problems:

(i) Monopole field

A magnetic field of the shape given in Figure 2.6 is represented by

$$\boldsymbol{B} = B_0 l_0^2 (r^2 + z^2)^{-\frac{3}{2}} \cdot (r, 0, z), \tag{2.95}$$

where B_0 and l_0 are constants. The singular point $r = 0$, towards which

the field lines are seen to converge, is excluded from the present discussion. The vector potential has only a component in the φ direction and can be written as

$$rA_{\varphi} = B_0 l_0^2 \, [1 - z/(r^2 + z^2)^{\frac{1}{2}}], \tag{2.96}$$

as is easily seen from equations (2.91) and (2.95). The particle motion in this field was first studied by POINCARÉ [1896] who found that the particle moves along a geodetic line.

The radial and axial components of the equation of motion (2.36) are

$$m\ddot{r} = mr\dot{\varphi}^2 + qr\dot{\varphi}B_z \tag{2.97}$$

and

$$m\ddot{z} = -\, qr\dot{\varphi}B_r. \tag{2.98}$$

After combination with equations (2.94), (2.95) and (2.96) these relations obtain the forms

$$m\ddot{r} = (1/mr)\{p_{\varphi 0} - qB_0 l_0^2[1 - z/(r^2 + z^2)^{\frac{1}{2}}]\}$$
$$\cdot \{r^{-2}[p_{\varphi 0} - qB_0 l_0^2[1 - z/\,(r^2 + z^2)^{\frac{1}{2}}]] + qB_0 l_0^2 z/(r^2 + z^2)^{\frac{3}{2}}\}, \tag{2.99}$$

$$m\ddot{z} = -\, qB_0 l_0^2 r\{p_{\varphi 0} - qB_0 l_0^2[1 - z/(r^2 + z^2)^{\frac{1}{2}}]\} \,/\, mr(r^2 + z^2)^{\frac{3}{2}} \tag{2.100}$$

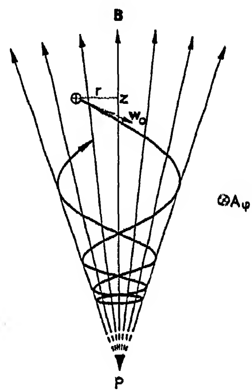


Fig. 2.6. Orbit in monopole field.

By direct insertion is easily seen that

$$z = k_0 r, \qquad rA_{\varphi} = r_0 A_{\varphi 0}, \qquad rw_{\varphi} = r_0 w_{\varphi 0} \tag{2.101}$$

satisfies (2.99) and (2.100) where k_0 is a constant which is determined by

$$k_0(1 + k_0^2)^{\frac{1}{2}} = - qB_0 l_0^2 / m r_0 w_{\varphi 0}. \tag{2.102}$$

Thus, the particle moves on a conical surface given by (2.101). The resulting equation for r becomes

$$r^3 \ddot{r} = \frac{r_0^2 w_{\varphi 0}^2}{1 + k_0^2} \equiv c_0 > 0. \tag{2.103}$$

Relations (2.102) and (2.103) can also be derived directly from a balance between the components of the centrifugal force $m w_{\varphi 0}^2 / r$ and all additional forces which act in the directions along and across $\boldsymbol{B}$. According to KAMKE [1948] the general solution of equation (2.103) is given by

$$c_1 r^2 = - c_0 - (c_1 t - c_2)^2 \text{ and } r^2 = 2t(- c_0)^{\frac{1}{2}} + c_3, \tag{2.104}$$

where c_1, c_2 and c_3 are constants of integration. The real solution which satisfies the initial conditions originates from the first of expressions (2.104) and becomes

$$r^2 = \frac{r_0^2 \, \{c_0 + [(c_0 + r_0^2 w_{\varphi 0}^2) t / r_0^2 + r_0 w_{\varphi 0}]^2\}}{c_0 + r_0^2 w_{\varphi 0}^2}. \tag{2.105}$$

After insertion of c_0 from (2.103) we find that the particle can move towards the axis of symmetry only up to a minimum radial distance

$$r_{\min} = r_0 \left| w_{\varphi 0} \right| / w_0. \tag{2.106}$$

A sketch of the orbit is shown in Figure 2.6. When the particle approaches the strong field region in the vicinity of the "magnetic pole" P it becomes retarded and is reflected. This is one example of the magnetic "mirror" effect which acts on a particle in a converging magnetic field as discussed later in Ch. 6, § 2.1 and in Ch. 7, § 3.1.

In the monopole field the particle moves all the time on the same flux tube $r A_\varphi = r_0 A_{\varphi 0}$. It will be shown in Ch. 6, § 2.1 that this is a property of the special field (2.95) which does not apply to a general case.

(ii) Dipole field

The orbits in a dipole field are of special interest to the study of cosmic rays and of the polar aurora in the earth's magnetic field. Reviews of this problem have been given by STÖRMER [1955], ALFVÉN [1950], LÜST [1953] and VALLARTA [1961] among others. Here we shall only include a short discussion of forbidden regions.

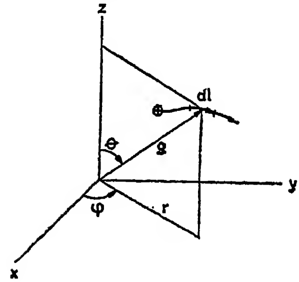


Fig. 2.7. Coordinates used to describe the particle motion.

Introduce a coordinate system (r, φ, z) as shown in Figure 2.7 with a magnetic dipole of moment $\boldsymbol{M}_p$ at the origin and directed in the positive z direction. The vector potential can then be written as

$$\boldsymbol{A} = (0, \mu_0 M_p r/4\pi\rho^3, 0). \tag{2.107}$$

Since the modulus w of the total velocity $\boldsymbol{w}$ is constant according to (2.92) we can use the arc length $l = wt$ along the particle path as independent variable instead of time. Conservation of momentum is now expressed by (2.94) in the form

$$r^2 \frac{d\varphi}{dl} + \frac{(q/|q|)c_{st}^2 r^2}{\rho^3} = \frac{p_{\varphi 0}}{m w_0} = \frac{r_0 w_{\varphi 0}}{w_0} + \frac{(q/|q|) r_0^2 c_{st}^2}{\rho_0^3}, \tag{2.108}$$

where

$$c_{st} = \left(\frac{\mu_0 |q| M_p}{4\pi m w_0}\right)^{\frac{1}{2}} \tag{2.109}$$

has the dimension of a length and is called Störmer's unit. The magnetic field enters only in c_{st}. At infinity the angular momentum $m(rw_\varphi)_\infty$ of the particle becomes equal to $p_{\varphi 0}$ because rA_φ vanishes when r tends to infinity. If q changes sign and the sign of φ is reversed at the same time, the equations are unchanged. We now restrict the discussion to $q > 0$.

Make the transformation of all length coordinates to dimensionless variables $r' = r/c_{st}$, $\rho' = \rho/c_{st}$ and $l' = l/c_{st}$. Expression (2.108) then reduces to

$$r'^2 \frac{d\varphi}{dl'} + \frac{r'^2}{\rho'^3} = \frac{r_0' w_{\varphi 0}}{w_0} + \frac{r_0'^2}{\rho_0'^3} \equiv 2\gamma'. \tag{2.110}$$

The result does not contain the physical constants M_p, m, q and w_0. It is

thus sufficient to determine the orbits given by this relation. All other trajectories can be found immediately from a simple scaling of r, ρ and l. This is also seen from the discussion of § 2.2; the Störmer unit is connected with the characteristic parameter k_1 of equation (2.43). For a given solution (r', ρ') the linear dimensions (r, ρ) of the orbit decrease with c_{st}. This is expected since c_{st} decreases when the radius of gyration increases.

The *forbidden regions* of the particle orbit can be obtained from (2.110). Introduce the angle θ in Figure 2.7, where $\cos\theta = r'/\rho'$. Consequently,

$$\rho' \cos\theta \frac{w_\varphi}{w} + \frac{\cos^2\theta}{\rho'} = 2\gamma'. \tag{2.111}$$

Since $|w_\varphi/w| \leqslant 1$ this gives the condition

$$\left| \frac{2\gamma'}{\rho' \cos\theta} - \frac{\cos\theta}{\rho'^2} \right| \leqslant 1 \tag{2.112}$$

for a particle to reach the point (r', φ', ρ') if it starts from $(r'_0, \varphi'_0, \rho'_0)$ with a given initial velocity. Points for which the inequality (2.112) does not hold cannot be reached by the particle and are situated in forbidden regions. For special illustrations to the result reference is made to STÖRMER [1955]. A general discussion of forbidden regions will be continued later in Ch. 7, § 2.

In this connexion should be mentioned that the particle orbits in a dipole field have been studied in a series of model experiments by BRUNBERG [1955, 1956] and BRUNBERG and DATTNER [1955]. Numerical calculations of such orbits are also described in a review by VALLARTA [1961].

(iii) Field from line current

HERTWECK [1959] has earlier investigated the particle motion in the field from a line current of strength I. Here the vector potential becomes

$$\boldsymbol{A} = (0,\ 0,\ A_0 \log(r/r_1)), \qquad A_0 = \mu_0 I/2\pi, \tag{2.113}$$

where r_1 is an arbitrary constant which does not affect the form of the corresponding magnetic field $\boldsymbol{B} = \operatorname{curl} \boldsymbol{A}$. In the z direction the equation of motion (2.36) gives

$$m\ddot{z} = q\dot{r}\frac{\partial A_z}{\partial r} = q\dot{r}A_0/r \tag{2.114}$$

and

$$\dot{z} = w_{z0} + (q/m)A_0 \log(r/r_0) \equiv (q/m)A_0 \log(r/r_2), \tag{2.115}$$

where r_0 indicates the starting point. This result is also immediately obtained from equations (2.51) and (2.59). Further, the conservation law (2.94) can be combined with (2.115) to

$$\dot{r}^2 = w_0^2 - r_0 w_{\varphi 0}/r^2 - (qA_0/m)^2 \log^2 (r/r_2). \tag{2.116}$$

The result can be used to calculate the forbidden regions, the border of which is obtained by putting $\dot{r}$ equal to zero. Two solutions $r = r_{\min}$ and $r = r_{\max}$ arise from this. Consequently, the particle will move in the space between two cylinders given by $r_{\min}$ and $r_{\max}$ (cf. also Ch. 7, § 2.1 (ii)). The time required for the particle to move from $r_{\min}$ to $r_{\max}$ and back to $r_{\min}$ is, by reasons of symmetry and according to equation (2.116),

$$\Delta t = 2 \int_{r_{\min}}^{r_{\max}} [w_0^2 - r_0 w_{\varphi 0}/r^2 - (qA_0/m)^2 \log^2 (r/r_2)]^{-\frac{1}{2}} \mathrm{d}r. \tag{2.117}$$

At the same time the particle drifts the distance

$$\Delta z = 2 \int_{r_{\min}}^{r_{\max}} (qA_0/m) \log (r/r_2) \cdot [w_0^2 - r_0 w_{\varphi 0}/r^2 - (qA_0/m)^2 \log^2 (r/r_2)]^{-\frac{1}{2}} \mathrm{d}r \tag{2.118}$$

in the z direction as calculated from (2.115) and (2.116).

The obtained results are of special interest because they admit an accurate computation of the mean drift velocity $\overline{w_z} = \Delta z/\Delta t$ in the z direction. They can be used to test the expressions for the drift in an inhomogeneous magnetic field according to the approximate methods described in Ch. 3, § 1. By making such a comparison Hertweck found the relative error in the approximation of $\overline{w_z}$ to be very small for small values of the ratio a/r between the radius of gyration a and the characteristic dimension $|B_\varphi/(\mathrm{d}B_\varphi/\mathrm{d}r)| = r$ of the magnetic field. For numerical results reference is made to the original work.

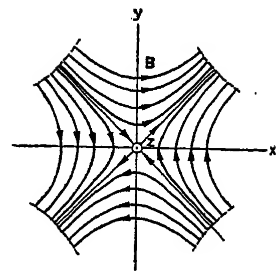
*4.3. HYPERBOLIC MAGNETIC FIELD

The particle orbits in the hyperbolic magnetostatic field of Figure 2.8 were earlier considered by ÅSTRÖM [1956] in a relativistic case. Here we shall present part of the results for a non-relativistic particle.

The magnetic field is given by

$$\boldsymbol{B} = 2(m/q)c_0(y, x, 0), \tag{2.119}$$

where c_0/q is a positive constant and a rectangular coordinate system has



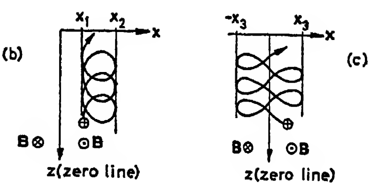


Fig. 2.8. Motion in hyperbolic magnetic field (cf. ÅSTRÖM [1956]). a. Shape of field. b. Example of orbit in case (i). There are two values of x^2 for which $\dot{x}$ vanishes according to equation (2.114). c. Example of orbit in case (ii). There is only one value of x^2 for which $\dot{x}$ vanishes.

been introduced. The equation of motion is

$$\dot{\boldsymbol{w}} = 2c_0(-xw_z, yw_x, xw_x - yw_y). \tag{2.120}$$

Due to the symmetry with respect to the z direction the z component of this equation is immediately integrated to

$$w_z = c_0(x^2 - y^2) - c_1, \quad c_1 = c_0(x_0^2 - y_0^2) - w_{z0}. \tag{2.121}$$

Since

$$(\boldsymbol{B} \cdot \nabla)w_z = 2c_0\boldsymbol{B} \cdot (x, -y, 0) = 0 \tag{2.122}$$

according to equation (2.119), w_z becomes constant on a magnetic field line. From the energy theorem $w^2 = w_0^2$ and together with (2.121) this results in

$$0 \leqslant w_x^2 + w_y^2 = w_0^2 - w_z^2 = [w_0 - c_0(x^2 - y^2) + c_1] \cdot [w_0 + c_0(x^2 - y^2) - c_1], \tag{2.123}$$

which can be used to determine the forbidden regions. Consequently, the border between the allowed and forbidden regions is given by the two hyperbolas

$$x^2 - y^2 = (c_1 \pm w_0)/c_0. \tag{2.124}$$

We now limit ourselves to a motion which only takes place in the plane $y = 0$ so that $w_y = 0$. For the points where also w_x vanishes the condition

$$x = \pm (c_1 \pm w_0)^{\frac{1}{2}}/c_0^{\frac{1}{2}} \tag{2.125}$$

has to be satisfied, where the four signs should be chosen independently of each other. There are three possibilities according to (2.125). Consider positive values of q:

(i) Four real roots

Then, $|c_1| > w_0$ and $c_1 > 0$. Introduce the parameters

$$c_2 = [2w_0/(w_0 + c_1)]^{\frac{1}{2}}, \quad c_3^2 = 1 - c_2^2 \tag{2.126}$$

and the new dimensionless variables

$$x' = c_2(c_0/2w_0)^{\frac{1}{2}}x, \quad t' = (2c_0w_0)^{\frac{1}{2}}t/c_2. \tag{2.127}$$

For $w_y = 0$ equation (2.123) yields

$$\left(\frac{\mathrm{d}x'}{\mathrm{d}t'}\right)^2 = (1 - x'^2)(x'^2 - c_3^2). \tag{2.128}$$

According to JAHNKE and EMDE [1945] the solution becomes

$$x' = \mathrm{dn}\, t' \tag{2.129}$$

with the elliptic function dn t'. Further, from equation (2.121)

$$\dot{z} = w_0 + 2w_0(\mathrm{dn}^2 t' - 1)/c_2^2. \tag{2.130}$$

After having introduced $\mathrm{d}z/\mathrm{d}t'$ instead of $\dot{z}$ this relation can be integrated to

$$z = \left(\frac{2w_0}{c_0}\right)^{\frac{1}{2}} \cdot \frac{\mathrm{zn}\, t' - c_2^4 \mathrm{C}\, t'/2\,\mathrm{K}}{c_2}, \tag{2.131}$$

where the Jacobian Zeta-function zn t' is periodic with the period 2K, and C and K are elliptic integrals defined by Jahnke and Emde among others.

(ii) Two real and two imaginary roots

With the new variables x' and $t'' = (2c_0w_0)^{\frac{1}{2}}$ equation (2.123) now becomes

$$\left(\frac{\mathrm{d}x'}{\mathrm{d}t''}\right)^2 = (1 - x'^2)(c_5^2 + c_4^2\, t'^2), \tag{2.132}$$

where $c_4 = 1/c_2$ and $c_5^2 = 1 - c_4^2$. The solution is the elliptic function

$$x' = \operatorname{cn} t''. \tag{2.133}$$

From (2.121) we now obtain after integration

$$z = \frac{w_0}{2c_0} \cdot \left[2 \operatorname{zn} t'' + \frac{(2\mathrm{E} - \mathrm{K})t''}{\mathrm{K}} \right], \tag{2.134}$$

where E is an elliptic integral defined by Jahnke and Emde among others.

(iii) No real roots

This case is not of physical interest since the velocity becomes purely imaginary.

In case (i) the path resembles that of a trochoid as shown in Figure 2.8b. In case (ii) the path crosses the zero line at the centre of Figure 2.8a, where the magnetic field vanishes and the crossing points are centra of symmetry of the path such as in Figure 2.8c. The magnetic field has opposite directions at opposite sides of the zero line. When the particle crosses the line in case (ii) the orbit passes an inflexion point and the centre of curvature jumps from one side of the orbit to the other. For a detailed description of the orbits reference is made to the original paper. The orbits are of special interest in certain types of electronic tubes.

*4.4. HOMOGENEOUS TIME-DEPENDENT MAGNETIC FIELD

Consider a homogeneous magnetic field $\boldsymbol{B} = (0, 0, B(t))$ which varies in time and is directed along the z axis of a rectangular coordinate system. The induction law (2.1) then becomes

$$\operatorname{curl} \boldsymbol{E} = - \hat{\boldsymbol{z}} \dot{B}, \tag{2.135}$$

where $\hat{\boldsymbol{z}}$ is the unit vector along the z axis. Further assume no space charge σ to be present, i.e., $\operatorname{div} \boldsymbol{E} = 0$ according to (2.7). This condition and (2.135) are satisfied by

$$\boldsymbol{E} = - \tfrac{1}{2}(\hat{\boldsymbol{z}} \times \rho)\dot{B}, \tag{2.136}$$

where ρ is the position vector. Introduce the gyro frequency $\omega_g(t) = qB(t)/m$ and the equation of motion (2.36) can be written as

$$\ddot{\rho} + \tfrac{1}{2}(\hat{\boldsymbol{z}} \times \rho)\dot{\omega}_g + \hat{\boldsymbol{z}} \times \rho\omega_g = 0. \tag{2.137}$$

With the substitution $\zeta_1 = x + iy$ and $i \equiv (-1)^{\frac{1}{2}}$ the x and y components

of (2.137) can be combined to

$$\ddot{\zeta}_1 + i\omega_g\dot{\zeta}_1 + \tfrac{1}{2}i\dot{\omega}_g\zeta_1 = 0. \tag{2.138}$$

By means of the additional substitution

$$\zeta = \zeta_1 \exp\left(-\tfrac{1}{2}i \int \omega_g \mathrm{d}t\right) \tag{2.139}$$

equation (2.138) is transformed into

$$\ddot{\zeta} + \tfrac{1}{4}\omega_g^2\zeta = 0, \tag{2.140}$$

as earlier shown by HERTWECK and SCHLÜTER [1957]. This equation is identical with that of a harmonic oscillator with a restoring force that varies in time. A discussion of such an oscillator will be performed later in Ch. 4, § 2 in terms of approximate methods.

The solution of (2.140) for an arbitrary function $\omega_g(t)$ leads to involved calculations. Here we shall only consider some simple special cases:

(i) $\omega_g = \omega_{g0} \cos \omega_0 t$; ω_0 and ω_{g0} constant

Equation (2.140) then has the form

$$\frac{\mathrm{d}^2\zeta}{\mathrm{d}t'^2} + \frac{\omega_{g0}^2}{8\omega_0^2}(1 + \cos 2t')\zeta = 0 \tag{2.141}$$

if we put $t' = \omega_0 t$. Equation (2.141) is a Mathieu differential equation which has been frequently discussed in the mathematical literature.

(ii) $\omega_g = \omega_{g0}(t/t_0)^{\frac{1}{2}\nu}$ where ν is an integer and t_0 constant

Equation (2.140) becomes

$$\frac{\mathrm{d}^2\zeta}{\mathrm{d}t'^2} + c_0 t'^{\nu}\zeta = 0, \tag{2.142}$$

where $t' = t/t_0$ and $c_0 = \frac{1}{4}(\omega_{g0}t_0)^2$. According to JAHNKE and EMDE [1945] the solution is

$$\zeta = t'^{\frac{1}{2}} Z_{1/(\nu+2)}\left[2c_0^{\frac{1}{2}}(\nu+2)^{-1}\, t'^{(1+\frac{1}{2}\nu)}\right], \tag{2.143}$$

where $Z_{1/(\nu+2)} = c_1 J_{1/(\nu+2)} + c_2 N_{1/(\nu+2)}$ is a linear combination of the Bessel functions $J_{1/(\nu+2)}$ and $N_{1/(\nu+2)}$ with the real or complex constants c_1 and c_2 of integration.

ORBIT THEORY

There are comparatively few cases where the exact solution can be found to the motion of a charged particle in a magnetic field. Already for a dipole field the situation becomes rather complicated as is readily demonstrated by the trajectories found by STÖRMER [1907, 1930, 1934, 1936, 1955]. A detailed analysis of the orbits of a charged particle is of special value e.g. in applications to cosmic ray physics. They are necessary whenever the particle energy is high enough for the radius of curvature of the orbit to become comparable to the characteristic dimensions of the magnetic and electric fields.

On the other hand, in some of the most important applications to cosmical physics and to plasma experiments the particles are immersed in very strong magnetic fields which reduce the radius of gyration to quite small values. The essential features of the motion can then be described by a circle-like gyration superimposed on the drift of a guiding centre. In such cases it does not become necessary to determine the detailed structure of the particle orbit. A first order *perturbation theory* on the guiding centre motion was developed by ALFVÉN [1940, 1950] who also discussed the limitations of the involved approximations. The theory provides simple methods for detailed calculations of the mean particle motion and also simplifies its physical interpretation. Reviews and discussions of this subject have been presented by SPITZER [1952, 1956], ALLIS [1956], LINHART [1960], CHANDRASEKHAR and TREHAN [1960], ROSE and CLARK [1961], FERRARO and PLUMPTON [1961], and ALFVÉN and FÄLTHAMMAR [1963]. Recently the results have been generalized by HELLWIG [1955] and VANDERVOORT [1960] to include relativistic effects (see Ch. 9, § 4). These authors as well as BOGULJUBOV and ZUBAREV [1955], KRUSKAL [1957], BRINKMAN [1959] and NORTHROP [1960, 1961] also discuss higher order approximations of the perturbation theory.

1. Motion of the Guiding Centre

Before a detailed study of the particle orbits can be undertaken, we have to specify the starting points of the perturbation theory as well as the signif-

icance of introduced approximations. Assume with ALFVÉN [1940, 1950] that the magnetic field variations in space and time are small within a Larmor radius a and a Larmor period $2\pi/\omega_g$, i.e.,

$$a\left|\frac{\partial B_j}{\partial x_k}\right| / |B_j| \ll 1 \tag{3.1}$$

and

$$\frac{1}{\omega_g}\left|\frac{dB_j}{dt}\right| / |B_j| \ll 1. \tag{3.2}$$

This should apply to all components B_j of the magnetic field and to all space coordinates x_k. The time derivative in equation (3.2) refers to a frame of reference which follows the particle. We further assume relations equivalent to (3.1) and (3.2) to be valid for any external force field $\boldsymbol{F}$ which interacts with the particle.

According to conditions (3.1) and (3.2) the particle should experience only small relative variations in $\boldsymbol{B}$ and $\boldsymbol{F}$ during a Larmor period. One of the consequences of this is that we have to avoid excessively large longitudinal velocities. Therefore, the longitudinal force $\boldsymbol{F}_{\parallel} = \boldsymbol{F} \cdot \boldsymbol{B}/B$ and the corresponding acceleration should remain sufficiently small (cf. KRUSKAL [1958]). However, this should not be interpreted in the way that $\boldsymbol{F}_{\parallel}$ must vanish identically. In certain cases the magnetic field inhomogeneity may namely produce effects which partly balance that of $\boldsymbol{F}_{\parallel}$ and by which the longitudinal velocity becomes limited even in presence of a longitudinal force field (cf. ALFVÉN and FÄLTHAMMAR [1963] and Ch. 7, § 3.1).

We also have to impose a restriction on the transverse force $\boldsymbol{F}_{\perp} = \boldsymbol{F} - \boldsymbol{F}_{\parallel}$. Thus, the latter should not be allowed to give rise to a transverse particle velocity and a Larmor radius which become large enough to invalidate conditions (3.1) and (3.2). A further discussion of the restrictions imposed on $\boldsymbol{F}_{\parallel}$ and $\boldsymbol{F}_{\perp}$ by the perturbation theory will be made in § 1.3 of this chapter.

Finally, the reactions of the individual particle on the fields $\boldsymbol{B}$ and $\boldsymbol{F}$ are neglected and the latter are treated as given constraints.

With these basic assumptions we now write down the equation of motion (2.36) in the form

$$\varepsilon \frac{d\boldsymbol{w}}{dt} = \frac{\boldsymbol{F}}{q} + \boldsymbol{w} \times \boldsymbol{B}, \quad \varepsilon = \frac{m}{q}, \quad \boldsymbol{w} = \frac{d\boldsymbol{\rho}}{dt}. \tag{3.3}$$

The left hand member of this relation and the last term of its right hand

member have the same forms as the corresponding terms of the dimensionless equation (2.42), where the parameter k_1 corresponds to ε.

There are at least two ways in which we can interpret the parameter ε. Firstly, choose the characteristic length L_c in equation (2.43) equal to that of the magnetic field and put t_c equal to the time during which the external fields are "seen" to vary appreciably, as observed in a frame following the particle. Then, k_1 and ε become equal to the ratio between the gyro period and this time. Secondly, put instead L_c equal to the Larmor radius and t_c equal to the gyro period. Then, k_1 and ε become equal to the ratio between the Larmor radius and the characteristic length of the magnetic field. In both cases we therefore see that ε is a small quantity when the magnetic field is very strong. Then, the gyro period becomes short compared to the characteristic times of the external fields and the Larmor radius is much less than the characteristic lengths of the same fields, as required by our basic assumptions (3.1) and (3.2). Of course, this does not imply that we can simply neglect the left hand member of equation (3.3); this is only justified in some special cases. Instead, a sequence of refined approximations to the real particle orbit will be obtained from an expansion in terms of the "smallness parameter" ε.

Now turn to a study of the particle orbit on the basis of equation (3.3). According to Ch. 2, § 4.1 (i) the orbit in a homogeneous magnetic field has the form of a helix and can be described by a velocity $\boldsymbol{W}_0$ of gyration superimposed on a motion of the guiding centre. The particle position is then given by

$$\boldsymbol{\rho}(t) = \boldsymbol{C}(t) + \varepsilon \boldsymbol{C}_1 \cos \omega_g t + \varepsilon \boldsymbol{S}_1 \sin \omega_g t = \boldsymbol{C} + \boldsymbol{a}, \tag{3.4}$$

where $\boldsymbol{C}(t)$ indicates the position of the guiding centre which "slides" along a field line. Further

$$\varepsilon \boldsymbol{C}_1 = \hat{\boldsymbol{B}} \times \boldsymbol{W}_0/\omega_g, \qquad \varepsilon \boldsymbol{S}_1 = \boldsymbol{W}_0/\omega_g \tag{3.5}$$

are two perpendicular vectors, each of modulus a.

Starting from these results we now make the transition to slightly inhomogeneous fields $\boldsymbol{B}$ and $\boldsymbol{F}$ which satisfy conditions (3.1) and (3.2) and the assumptions made at the beginning of this paragraph. The path will then deviate slightly from that given by the solutions (3.4) and (3.5). We therefore try an expansion for the position vector as suggested by BOGULJUBOV and ZUBAREV [1955] and by KRUSKAL [1957] (see also BRINKMAN [1959]):

$$\begin{aligned}\boldsymbol{\rho}(t) &= \boldsymbol{C}(t) + \sum_{\nu=1}^{\infty} \varepsilon^{\nu}\{\boldsymbol{C}_{\nu}(t) \cos [\nu\vartheta(t)/\varepsilon] + \boldsymbol{S}_{\nu}(t) \sin [\nu\vartheta(t)/\varepsilon]\} \\ &\equiv \boldsymbol{C}(t) + \boldsymbol{a}(t).\end{aligned} \tag{3.6}$$

Here $\mathrm{d}\boldsymbol{C}/\mathrm{d}t$, $\boldsymbol{C}_\nu(t)$ and $\boldsymbol{S}_\nu(t)$ are slowly varying functions of time. The gyration is associated with ϑ/ε which is a rapidly fluctuating periodic function due to the smallness of ε.

BERKOWITZ and GARDNER [1959] have shown that the expansion (3.6) is a correct asymptotic series for the true particle trajectory. In other words, this implies that the form (3.6) can be used for the exact solution of the equation (3.3) of motion.

In a homogeneous magnetic field and with $\boldsymbol{F} = 0$ expression (3.6) should approach eq. (3.4) in such a way that $\boldsymbol{C}_\nu$ and $\boldsymbol{S}_\nu$ disappear for $\nu > 1$ and ϑ becomes equal to Bt. For inhomogeneous fields, $\boldsymbol{B}$ and $\boldsymbol{F}$, the expansion (3.6) implies that the particle should still perform a non-oscillating drift given by $\boldsymbol{C}(t)$ superimposed on an oscillating motion described by $\boldsymbol{a}(t)$. The vector $\boldsymbol{a}(t)$ connects the particle with the centre $\boldsymbol{C}(t)$ of gyration. To lowest order ($\nu = 1$) the former represents a gyration around the field lines at the gyro frequency. To the motion which results from the first order solution of $\boldsymbol{a}$ are added some small corrections which originate from the higher order terms of the expansion (3.6). These terms oscillate at multiples of the gyro frequency and with amplitudes which are at least of order ε compared to the modulus of $\boldsymbol{a}$.

So far we have not specified exactly what is meant by the radius of gyration in higher order. As a matter of fact, the coefficients of equation (3.6) are determined first when the expression for $\boldsymbol{\rho}$ is substituted into the equation (3.3) of motion and the latter is being solved. The coefficients will then contain the given fields $\boldsymbol{B}$ and $\boldsymbol{F}$. Whenever we discuss orbit theory in higher order in this volume we shall use $\boldsymbol{a}$ for the total oscillating part of the particle motion as defined in (3.6). We also remember that $|\,\boldsymbol{a}\,|$ in first order becomes identical with the expression (2.81) for the Larmor radius.

To be able to solve (3.3) we have to use expressions for the fields $\boldsymbol{B}$ and $\boldsymbol{F}$ in terms of the particle position. Therefore expand the fields in Taylor series around the centre $\boldsymbol{C}$ of gyration;

$$\boldsymbol{B}[\boldsymbol{\rho}(t)] = \boldsymbol{B}_\mathrm{C} + \sum_{\nu=1}^{\infty} \frac{1}{\nu!} (\boldsymbol{a} \cdot \nabla)^\nu \boldsymbol{B}_\mathrm{C} \tag{3.7}$$

and

$$\boldsymbol{F}[\boldsymbol{\rho}(t)] = \boldsymbol{F}_\mathrm{C} + \sum_{\nu=1}^{\infty} \frac{1}{\nu!} (\boldsymbol{a} \cdot \nabla)^\nu \boldsymbol{F}_\mathrm{C}, \tag{3.8}$$

where subscript ($_\mathrm{C}$) indicates quantities evaluated at the instantaneous centre $\boldsymbol{C}(t)$ of gyration.

With the present starting points and assumptions we are now in a position to study the guiding centre drift in detail.

1.1. FIRST ORDER ORBIT THEORY

In a first approach to our problem we neglect higher order effects and shall, on somewhat intuitive grounds, determine the first order equation of motion of the guiding centre. The approximations which are used in this connexion will be further examined in § 1.3. Introduce from equation (3.6)

$$\boldsymbol{u} = \mathrm{d}\boldsymbol{C}/\mathrm{d}t, \qquad \boldsymbol{W} = \mathrm{d}\boldsymbol{a}/\mathrm{d}t, \qquad \boldsymbol{w} = \boldsymbol{u} + \boldsymbol{W}. \tag{3.9}$$

Here the total velocity $\boldsymbol{w}$ of the particle is divided into a velocity $\boldsymbol{u}$ of the guiding centre and a velocity $\boldsymbol{W}$ of gyration (Larmor motion). The latter is perpendicular to $\boldsymbol{B}$ in first order. When the notations of (3.9) are substituted into (3.3) we obtain

$$m\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} - \boldsymbol{F}_{\mathrm{C}} - q\boldsymbol{u} \times \boldsymbol{B}_{\mathrm{C}} - q\boldsymbol{W} \times \Delta\boldsymbol{B} = q\boldsymbol{u} \times \Delta\boldsymbol{B} + q\boldsymbol{W} \times \boldsymbol{B}_{\mathrm{C}} - m\frac{\mathrm{d}\boldsymbol{W}}{\mathrm{d}t} + \Delta\boldsymbol{F}. \tag{3.10}$$

Expressions

$$\Delta\boldsymbol{B} = \boldsymbol{B}(\rho) - \boldsymbol{B}_{\mathrm{C}} \approx (\boldsymbol{a}\cdot\nabla)\boldsymbol{B}_{\mathrm{C}}\ , \quad \Delta\boldsymbol{F} = \boldsymbol{F}(\rho) - \boldsymbol{F}_{\mathrm{C}} \approx (\boldsymbol{a}\cdot\nabla)\boldsymbol{F}_{\mathrm{C}} \tag{3.11}$$

indicate the deviations of $\boldsymbol{B}$ and $\boldsymbol{F}$ from the values $\boldsymbol{B}_{\mathrm{C}}$ and $\boldsymbol{F}_{\mathrm{C}}$ at the instantaneous guiding centre in terms of the derivatives at the same point. The magnitude of these deviations is determined by the field variations over the Larmor radius a.

The variations of $\boldsymbol{B}$ and the influence of $\boldsymbol{F}$ perturb the gyration only very little. As a first approximation we therefore represent the oscillating motion by the expressions (2.80) – (2.82) for $\boldsymbol{a}$ and $\boldsymbol{W}$ in a homogeneous magnetic field. The second and third terms of the right hand member of equation (3.10) then nearly cancel. Their difference is balanced by small oscillating contributions from the rest of the same member as well as from the last term of the left hand member of equation (3.10).

As a matter of fact, a detailed analysis of these oscillations is not needed here because we are merely interested in the mean behaviour of the particle orbit during a gyro period $2\pi/\omega_{\mathrm{g}}$. Therefore, we now multiply (3.10) by $(\omega_{\mathrm{g}}/2\pi)\mathrm{d}t$ and integrate over a gyro period. Introduce $\langle\chi\rangle$ to indicate mean values of any quantity χ during such a period. Then, equation (3.10) re-

duces to

$$m\frac{d\boldsymbol{u}}{dt} = \boldsymbol{F}_C + q\boldsymbol{u} \times \boldsymbol{B}_C + q\langle \boldsymbol{W} \times \Delta\boldsymbol{B}\rangle. \tag{3.12}$$

In the deduction of this result we have made use of the fact that $d\boldsymbol{C}/dt = \boldsymbol{u} \approx \langle \boldsymbol{u}\rangle$ is a slowly varying function during a gyro period, and that $\langle a_k\rangle = 0$, $\langle W_k\rangle = 0$ due to the sinusoidal time variation of the components of $\boldsymbol{a}$ and $\boldsymbol{W}$ in first order. Disregarding the last term of (3.12), which contains the magnetic field inhomogeneity, the result looks at a first sight equivalent to the equation (2.36) of motion. However, equation (2.36) concerns the detailed, oscillating particle motion at velocity $\boldsymbol{w}$, whereas (3.12) represents the motion of a kind of equivalent particle. The latter is connected with the centre of gyration which changes its velocity $\boldsymbol{u}$ only little during a Larmor period.

Already at this stage the effect of the magnetic field inhomogeneity on the drift motion can be analysed by some simple physical arguments applied to the last term of equation (3.12). As seen from a coordinate system which follows the particle, both $\boldsymbol{W}$ and $\Delta\boldsymbol{B}$ oscillate with the gyro frequency and there is a net contribution from their vector product during a gyro period.

When the modulus of the magnetic field strength has a gradient ∇B along $\boldsymbol{B}$ the field lines will converge and a force $q\boldsymbol{W} \times \Delta\boldsymbol{B}$ arises as shown in Figure 3.1. This force tends to repel the particle from regions of high field

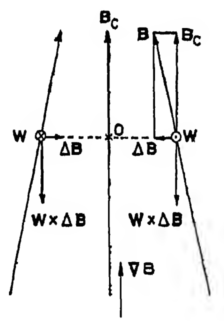


Fig. 3.1. Forces acting on a particle moving in an inhomogeneous magnetic field with ∇B along $\boldsymbol{B}$.

strength and is the basic mechanism of a magnetic "mirror" as discussed in Ch. 6, § 2.1 and in Ch. 7, § 3.1. It is also connected with the diamagnetic properties of an ionized gas.

When the magnetic field gradient ∇B has a component across $\boldsymbol{B}$ the force $q\boldsymbol{W} \times \Delta\boldsymbol{B}$ due to the inhomogeneity fluctuates during a gyro period as sketched in Figure 3.2a. There is a mean contribution directed upwards in

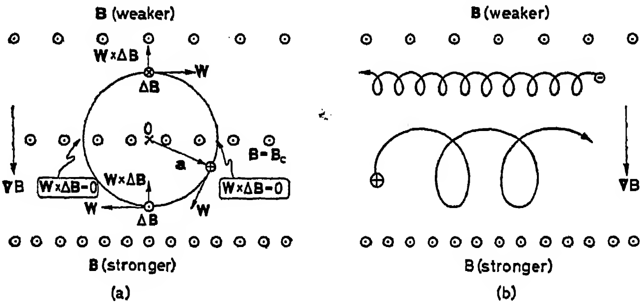


Fig. 3.2. Forces acting on a particle when ∇B is perpendicular to $\boldsymbol{B}$. a. The magnetic field inhomogeneity $\Delta\boldsymbol{B}$ produces a mean force directed upwards in the figure for a positive ion. b. The drift motion due to the inhomogeneity has opposite directions for ions and electrons.

the figure for a positive ion. This corresponds to an equivalent force which, in analogy with the electric field drift of Ch. 2, § 4.1 (ii) and Figure 2.4b, produces a drift motion to the right as demonstrated in Figure 3.2b. This motion is also easily understood from the latter figure. Thus, the total velocity $\boldsymbol{w}$ remains constant and the radius of curvature is smaller for the lower parts of the orbits, where the field is stronger, than for the upper parts, where it is weaker. Ions and electrons drift in opposite directions.

According to equations (2.80) and (2.82) $\boldsymbol{a}$ and $\boldsymbol{W}$ are perpendicular to $\boldsymbol{B}$ in first order. We introduce a local rectangular coordinate system with the z axis along $\boldsymbol{B}$. Then, equations (2.82) and (3.11) yield

$$\begin{aligned} \boldsymbol{W} \times \Delta\boldsymbol{B} &= \omega_g(\boldsymbol{a} \times \hat{\boldsymbol{B}}) \times [(\boldsymbol{a}\cdot\nabla)\boldsymbol{B}] \\ &\approx \omega_g(-a_x(\boldsymbol{a}\cdot\nabla)B_z,\ -a_y(\boldsymbol{a}\cdot\nabla)B_z,\ a_y(\boldsymbol{a}\cdot\nabla)B_y + a_x(\boldsymbol{a}\cdot\nabla)B_x), \end{aligned} \tag{3.13}$$

where subscript ($_c$) can be dropped since $|\Delta\boldsymbol{B}|/B \ll 1$. This implies that we assume the first derivatives of $\boldsymbol{B}$ to be constant within the Larmor radius and neglect second order terms in the expansion (3.7). When mean values are formed over a gyro period we observe that $\langle a_x a_y\rangle = 0$ and $\langle a_x^2\rangle = \langle a_y^2\rangle = \frac{1}{2}a^2 = W^2/2\omega_g^2$ according to (2.80). Consequently,

$$\langle \boldsymbol{W} \times \Delta\boldsymbol{B}\rangle = -(W^2/2\omega_g)\nabla B_z, \tag{3.14}$$

where B_x and B_y are expressed in terms of B_z by means of the condition div $\boldsymbol{B} = 0$. Now

$$\nabla B_z = \left(\frac{1}{2B_z}\right) \nabla B_z{}^2 = \frac{B\nabla B - B_x \nabla B_x - B_y \nabla B_y}{B_z} \approx \nabla B. \tag{3.15}$$

This result holds because B_x and B_y are much less than B within a radius of gyration where the curvature of the field is very small. Introduce the modulus $M = mW^2/2B$ of the equivalent magnetic moment given by (2.83). According to equations (3.14) and (3.15) the equation of motion (3.12) of the guiding centre then becomes

$$m\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \boldsymbol{F} + q\boldsymbol{u} \times \boldsymbol{B} - M\nabla B, \quad M = \frac{mW^2}{2B} \tag{3.16}$$

when subscript ($_c$) is dropped. This can be done on account of the smallness in the variations of $\boldsymbol{F}$ and $\boldsymbol{B}$ across a Larmor orbit. Observe that the total time derivative of the left hand member of (3.16) refers to a coordinate system following the particle. However, since the variations of $\boldsymbol{u}$ across a Larmor orbit become very small this derivative is approximately equal to that in a system which follows the guiding centre motion. Equation (3.16) will be rederived and verified in § 1.3 by means of the expansion (3.6), as demonstrated by equation (3.40).

The result (3.16) implies that the mean orbit is described by an equivalent particle which moves with the velocity $\boldsymbol{u}$ of the guiding centre and has an equivalent magnetic moment $\boldsymbol{M} = -M\boldsymbol{B}/B$. The motion can be resolved into components along and across the magnetic field. Scalar multiplication of equation (3.16) by the unit vector $\hat{\boldsymbol{B}} = \boldsymbol{B}/B$ yields for the longitudinal motion

$$m\hat{\boldsymbol{B}} \cdot \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \boldsymbol{F} \cdot \hat{\boldsymbol{B}} - M(\hat{\boldsymbol{B}} \cdot \nabla)B. \tag{3.17}$$

From the vector product between $\boldsymbol{B}/qB^2$ and (3.16) we obtain the transverse drift velocity

$$\boldsymbol{u}_\perp = \left(\boldsymbol{F} - M\nabla B - m\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\right) \times \frac{\boldsymbol{B}}{qB^2}. \tag{3.18}$$

Part of the acceleration within the bracket of the present equation is due to the longitudinal drift $\boldsymbol{u}_\parallel = \boldsymbol{u} - \boldsymbol{u}_\perp$. It is produced by the centrifugal force which originates from the radius of curvature $\boldsymbol{R}$ of the magnetic field lines as indicated in Figure 3.3. The acceleration can be written as

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = u_\parallel \frac{\mathrm{d}\hat{\boldsymbol{u}}_\parallel}{\mathrm{d}t} + \hat{\boldsymbol{u}}_\parallel \frac{\mathrm{d}u_\parallel}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{u}_\perp}{\mathrm{d}t}. \tag{3.19}$$

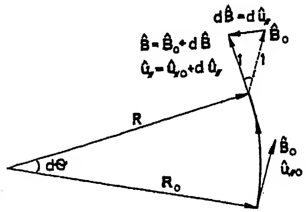


Fig. 3.3. The longitudinal drift motion $\boldsymbol{u}_{\parallel}$ produces a centrifugal acceleration in a magnetic field $\boldsymbol{B}$ with radius of curvature $\boldsymbol{R}$.

In a time interval $\mathrm{d}t$ the particle moves the distance $|\boldsymbol{R}|\,\mathrm{d}\theta = u_{\parallel}\mathrm{d}t$ along a field line from a point where the field strength is $\boldsymbol{B}_0$ to a point where it is $\boldsymbol{B}$. The unit vector $\hat{\boldsymbol{u}}_{\parallel} = \boldsymbol{u}_{\parallel}/u_{\parallel}$ changes by the amount

$$\begin{aligned}\mathrm{d}\hat{\boldsymbol{u}}_{\parallel} &= \hat{\boldsymbol{u}}_{\parallel} - \hat{\boldsymbol{u}}_{\parallel 0} = \hat{\boldsymbol{B}} - \hat{\boldsymbol{B}}_0 = (|\boldsymbol{R}|\,\mathrm{d}\theta\hat{\boldsymbol{B}}\cdot\nabla)\hat{\boldsymbol{B}} \\ &= u_{\parallel}(\hat{\boldsymbol{B}}\cdot\nabla)\hat{\boldsymbol{B}}\,\mathrm{d}t.\end{aligned} \tag{3.20}$$

From well known vector identities we obtain

$$(\hat{\boldsymbol{B}}\cdot\nabla)\hat{\boldsymbol{B}} = -\hat{\boldsymbol{B}}\times\operatorname{curl}\hat{\boldsymbol{B}} = (B\nabla_{\perp}B - \boldsymbol{B}\times\operatorname{curl}\boldsymbol{B})/B^2. \tag{3.21}$$

Introduction of expressions (3.19) – (3.21) into (3.18) results in

$$\boldsymbol{u}_{\perp} = \left[\boldsymbol{F} - M\left(1+\frac{2u_{\parallel}^2}{W^2}\right)\nabla B - m\,\frac{\mathrm{d}\boldsymbol{u}_{\perp}}{\mathrm{d}t}\right]\times\frac{\boldsymbol{B}}{qB^2} + \left(\frac{2Mu_{\parallel}^2}{W^2qB}\right)(\operatorname{curl}\boldsymbol{B})_{\perp}, \tag{3.22}$$

where the limit $\boldsymbol{B} = 0$ is excluded according to the basic assumptions of this paragraph.

The inertia term within the square bracket can be determined by an iteration process applied to equation (3.22). With $\boldsymbol{u}_m$ denoting the velocity given by (3.25) such a process implies that the expression for $\boldsymbol{u}_{\perp} - \boldsymbol{u}_m$ is substituted into the time derivative of equation (3.22). This yields a refined approximation to the velocity $\boldsymbol{u}_{\perp}$. The process can then be repeated to obtain more accurate solutions.

Among the possible sources of transverse drift motions included in (3.22) we notice the following:

$$\boldsymbol{u}_F = \boldsymbol{F}\times\boldsymbol{B}/qB^2 \qquad \text{(external force drift)}, \tag{3.23}$$

$$\boldsymbol{u}_B = \left[\frac{M(1+2u_{\parallel}^2/W^2)}{qB^2}\right]\boldsymbol{B}\times\nabla B \qquad \text{(magnetic gradient drift)}, \tag{3.24}$$

$$\boldsymbol{u}_m = \left(\frac{m}{qB^2}\right)\boldsymbol{B} \times \frac{\mathrm{d}\boldsymbol{u}_\perp}{\mathrm{d}t} \qquad \text{(transverse inertia drift)}, \quad (3.25)$$

$$\boldsymbol{u}_E = \boldsymbol{E} \times \boldsymbol{B}/B^2 \qquad \text{(electric drift)}, \quad (3.26)$$

$$\boldsymbol{u}_\mathrm{p} = \left(\frac{m}{qB^4}\right)\boldsymbol{B} \times \left(\frac{\partial \boldsymbol{E}}{\partial t} \times \boldsymbol{B}\right) = \left(\frac{m}{qB^2}\right)\frac{\partial \boldsymbol{E}_\perp}{\partial t} \qquad \text{(polarization drift)}. \quad (3.27)$$

The external force drift and the electric drift are obvious from the earlier discussions in Ch. 2, § 4.1 (ii) and of Figure 2.4b for a particle "falling" in a force field and across the magnetic field. The electric drift is also obvious from the relativistic point of view. There always exists a coordinate system moving with respect to the "laboratory system" at the velocity $\boldsymbol{u}_E$. In this system the electric field vanishes. Observe that the electric drift is independent of the polarity of the particle charge, but not the drift in a gravitation field.

When the force field or the electric field are derived only from potential gradients situated in planes perpendicular to $\boldsymbol{B}$ the drift motions will follow the corresponding equipotential surfaces.

The magnetic gradient drift is directed along the surfaces $B = \text{const.}$ as seen from equation (3.24). It consists of a first part given by the unity term within the bracket of the same equation. This part arises from the influence of the field inhomogeneity on the Larmor motion. As will be seen later in Ch. 4, § 1.2 the moment M is an approximate constant of the motion and the first part of the magnetic gradient drift is therefore explicitly determined by the values of $\boldsymbol{B}$ and W. The influence of the magnetic field curvature appears both in the second part of the magnetic gradient drift and in the last term of equation (3.22). Observe that it depends on the ratio $u_\parallel^2/W^2$ which is not a constant of the motion, but changes when the particle drifts along the magnetic field. When the field lines are straight the second part of $\boldsymbol{u}_B$ and the last term of (3.22) cancel.

The drift of a particle in an inhomogeneous magnetic field was already discussed by Thomson in 1906 for the special case of a field generated by a line current. By means of the first order perturbation theory ALFVÉN [1940, 1950] showed that the motion of the centre of gyration closely followed the mean motion of the particle as calculated by STÖRMER [1930, 1934, 1936] for a magnetic dipole field. HERTWECK [1959] estimated the accuracy of expression (3.24) for a particle moving in the field of a line current as shown in Ch. 2, § 4.2. (iii). It was found that very small errors are introduced by the perturbation theory in this particular case.

As seen from an observer following an accelerated motion of the guiding centre there should occur an inertia force and a corresponding drift (3.25). The latter is quite analogous to that produced by a gravitation field. An important part of this motion is given by the polarization drift of equation (3.27). It is due to an acceleration of the electric drift and requires a work to be performed on the particle by the electric field. This is clearly demonstrated by the simple example of Figure 2.4e. Here the particle "falls" across the electric equipotential surfaces while being accelerated in the negative y direction in a static magnetic field. We can also understand this from a balance between the acceleration work and the energy extracted from the electric field. The latter is provided by the polarization drift $\boldsymbol{u}_p$ and requires the condition

$$\tfrac{1}{2}m \frac{\partial}{\partial t}(u_E^2) = q\boldsymbol{u}_p \cdot \boldsymbol{E} \tag{3.28}$$

to be satisfied, provided that we restrict ourselves to small velocities for which $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}_E \approx 0$. Substitute $\boldsymbol{E}_\perp = \boldsymbol{B} \times \boldsymbol{u}_E$ into this equation and remember that $(\boldsymbol{u}_p \times \boldsymbol{B}) \cdot \boldsymbol{u}_E$ does not vanish. For $\boldsymbol{u}_p$ we then obtain an expression identical with equation (3.27).

We observe that the polarization drift has opposite directions for particles of different polarity. This generates a charge separation as demonstrated by the example of Figure 2.4e, where the paths of an ion and an electron diverge. The separation is of fundamental importance to the behaviour of a magnetized plasma. As we shall see later in § 2 it can be expressed in terms of an equivalent dielectric constant.

There are further effects which can be produced by acceleration of the transverse guiding centre motion. It should be observed that $d\boldsymbol{u}_\perp/dt = \partial\boldsymbol{u}_\perp/\partial t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}_\perp$ contains contributions in addition to that which gives rise to the polarization drift. To take a specific example, we may mention the drift due to a transverse force field $\boldsymbol{F}_\perp = q\boldsymbol{E}_\perp$ of zero order. Such a field arises e.g. in a plasma which rotates rapidly around the axis of a magnetic mirror field. The drift $\boldsymbol{u}_E$ at the radial distance r from the axis produces a centrifugal force and a corresponding inertia drift of magnitude $mu_E{}^2/qBr$. The latter may very well provide an important contribution to equation (3.22). In particular, when $\boldsymbol{u}_E$ reaches the same order of magnitude as the thermal velocity W, the orbits will be somewhat like that of Figure 2.4c. The inertia drift arising from $\boldsymbol{u}_E$ is then of the same importance as that arising from $\boldsymbol{u}_\parallel$, provided that the radial distance r and the radius of curvature R of

the field lines are comparable as well as u_E and $u_\parallel$. The acceleration due to the rotation can also be taken explicitly into account, by using a frame of reference which rotates at the velocity $\boldsymbol{u}_E$, as will be shown later in Ch. 7, § 2.2.

A further analysis of the contributions to the inertia drift has recently been performed by NORTHROP [1961], who also includes a time dependence of the transverse electric field which is comparable to the period of gyration.

*1.2. SCALING LAWS

We have earlier treated the scaling laws of the exact particle motion in Ch. 2, § 2.2. An analogous discussion can be made with respect to the motion of the guiding centre. With $\boldsymbol{C} = C_c \cdot \boldsymbol{C}'$ and $\boldsymbol{W} = W_c \cdot \boldsymbol{W}'$ we obtain

$$K_1 \frac{\mathrm{d}^2 \boldsymbol{C}'}{\mathrm{d}t'^2} = - K_2 \nabla' \phi' - K_3 \nabla' \phi'_g - K_4 \frac{\partial \boldsymbol{A}'}{\partial t'} + \frac{\mathrm{d}\boldsymbol{C}'}{\mathrm{d}t'} \times \mathrm{curl}' \boldsymbol{A}' - K_5 W'^2 (\mathrm{curl}' \boldsymbol{A}')^{-2} \nabla' (\mathrm{curl}' \boldsymbol{A}')^2, \tag{3.29}$$

where the dimensionless parameters are

$$K_1 = \frac{mL_c}{qA_c t_c} = k_1, \quad K_2 = \frac{t_c \phi_c}{A_c C_c},$$
$$K_3 = \frac{m t_c \phi_{gc}}{q A_c C_c}, \quad K_4 = \frac{L_c}{C_c}, \quad K_5 = \frac{m t_c W_c^2}{4 q A_c C_c}. \tag{3.30}$$

For a set of similar configurations, in the sense of Ch. 2, § 2.2, all these parameters should be kept constant. Keeping K_4 constant is trivial; it only implies that a certain distance of the guiding centre from the origin is scaled in the same proportion as the dimensions of the configuration as a whole. We can put $C_c = \text{const.}\ L_c$ without loss of generality and drop K_4.

We first observe that an exact scaling of the orbit requires all constants of (3.30) to be taken into account. In particular, put L_c and t_c equal to the Larmor radius and the gyro period, so that $W_c = \text{const.}\ L_c/t_c$ and $K_5 = \text{const.}\ K_4$. We then arrive at the same scaling conditions as earlier in Ch. 2, § 2.2 for the exact equation of motion. This is understandable since a high accuracy is required in the determination of the particle orbit as long as the inertia term in (3.29) is retained. Especially the constancy of K_1 and K_2 implies that A_c^2/ϕ_c should be constant as well as $(B_c L_c)^2/\phi_c$, where B_c is the characteristic magnetic field strength. BLOCK [1956] has pointed out that this makes the scaling from cosmical to experimental conditions very difficult.

As an example, the voltage in the surroundings of the earth and over a distance equal to the earth's radius is about 10^4 V. The terrestrial magnetic field has a strength of the order of 10^{-5} V · sec · m^{-2}. In a model experiment with a field of 1 V · sec · m^{-2} and dimensions of the order of 0.1 m we would then have to use voltages of the order of 10^{-2} V. Such values are far too small to be realized in the laboratory in discharges which have to be sustained at much higher voltages.

The influence of the inertia term in (3.29) is small, as far as the shape of the guiding centre orbits is concerned. This is seen if we now choose $L_c \approx C_c$ and put t_c equal to the characteristic time of the transverse motion so that $L_c/t_c \approx u_\perp$. When the relative variations in space and time of ϕ, ϕ_g and $\boldsymbol{A}$ are of the same order as $\boldsymbol{u} = \mathrm{d}\boldsymbol{C}/\mathrm{d}t$ all terms with primed quantities in (3.29) will be of order unity. Since K_1 at the same time is roughly equal to the ratio between the gyro period and the characteristic time of the transverse drift, the inertia term will give a very small contribution compared to other terms of equation (3.29). For an approximate scaling of the guiding centre orbits it is then sufficient to require that K_2, K_3 and K_5 are kept constant, provided that it is justified to include the electric field from the space charges of the plasma into the quantities ϕ', ϕ_c and K_2 of equations (3.29) and (3.30). Combination of the expressions for K_2 and K_5 then yields the condition that W_c^2/ϕ_c should be constant, as earlier found by BLOCK [1956]. In such an approximation the influence of the left hand member of (3.29) is not taken into account and neither the gyration of the particle, nor the inertia effects and associated polarization and charge separation phenomena are scaled in the same way as the guiding centre orbits.

1.3. HIGHER ORDER APPROXIMATIONS

So far we have discussed the first order orbit theory. We shall now return to (3.6) and examine the situation more closely in terms of higher approximations. Following KRUSKAL [1958] we substitute the expansions given by equations (3.6), (3.7) and (3.8) into the equation of motion (3.3). The result is a rather involved relation containing non-oscillating terms as well as a series of terms with different frequencies of oscillation ($\nu = 1, 2, \ldots$). Since the equation has to be satisfied for all times t we have to equate the coefficients for each mode separately. Thus, for non-oscillating terms we get an equation of motion of the guiding centre

$$\begin{aligned}\varepsilon\ddot{\boldsymbol{C}} = \boldsymbol{F}/q &+ \tfrac{1}{4}\varepsilon^2[(\boldsymbol{C}_1\cdot\boldsymbol{\nabla})^2 + (\boldsymbol{S}_1\cdot\boldsymbol{\nabla})^2]\boldsymbol{F}/q \\ &+ \dot{\boldsymbol{C}}\times\boldsymbol{B} + \tfrac{1}{4}\varepsilon^2\dot{\boldsymbol{C}}\times[(\boldsymbol{C}_1\cdot\boldsymbol{\nabla})^2 + (\boldsymbol{S}_1\cdot\boldsymbol{\nabla})^2]\boldsymbol{B} \\ &- \tfrac{1}{2}\varepsilon\dot{\vartheta}[\boldsymbol{C}_1\times(\boldsymbol{S}_1\cdot\boldsymbol{\nabla}) - \boldsymbol{S}_1\times(\boldsymbol{C}_1\cdot\boldsymbol{\nabla})]\boldsymbol{B} \\ &+ \tfrac{1}{2}\varepsilon^2[\dot{\boldsymbol{C}}_1\times(\boldsymbol{C}_1\cdot\boldsymbol{\nabla}) + \dot{\boldsymbol{S}}_1\times(\boldsymbol{S}_1\cdot\boldsymbol{\nabla})]\boldsymbol{B} + \mathrm{O}(\varepsilon^3).\end{aligned} \tag{3.31}$$

Here we omit subscript ($_C$) which indicates that the fields $\boldsymbol{F}$ and $\boldsymbol{B}$ and their derivatives are evaluated at the centre of gyration. A dot is used to denote time derivatives.

For terms with the coefficient $\cos(\vartheta/\varepsilon)$ we have

$$\begin{aligned}\dot{\vartheta}^2\boldsymbol{C}_1 + \dot{\vartheta}\boldsymbol{S}_1\times\boldsymbol{B} &= \varepsilon(2\dot{\vartheta}\dot{\boldsymbol{S}}_1 + \ddot{\vartheta}\boldsymbol{S}_1 - \boldsymbol{C}_1\times\boldsymbol{B}) \\ &\quad - \varepsilon\dot{\boldsymbol{C}}\times[(\boldsymbol{C}_1\cdot\boldsymbol{\nabla})\boldsymbol{B}] - \varepsilon(\boldsymbol{C}_1\cdot\boldsymbol{\nabla})\boldsymbol{F}/q + \mathrm{O}(\varepsilon^2)\end{aligned} \tag{3.32}$$

and for terms with $\sin(\vartheta/\varepsilon)$

$$\begin{aligned}\dot{\vartheta}^2\boldsymbol{S}_1 - \dot{\vartheta}\boldsymbol{C}_1\times\boldsymbol{B} &= -\varepsilon(2\dot{\vartheta}\dot{\boldsymbol{C}}_1 + \ddot{\vartheta}\boldsymbol{C}_1 + \dot{\boldsymbol{S}}_1\times\boldsymbol{B}) \\ &\quad - \varepsilon\dot{\boldsymbol{C}}\times[(\boldsymbol{S}_1\cdot\boldsymbol{\nabla})\boldsymbol{B}] - \varepsilon(\boldsymbol{S}_1\cdot\boldsymbol{\nabla})\boldsymbol{F}/q + \mathrm{O}(\varepsilon^2).\end{aligned} \tag{3.33}$$

Equations (3.32) and (3.33) are mainly connected with the gyration around the guiding centre. Terms corresponding to higher modes of oscillation ($\nu \geqslant 2$) will not be discussed here.

To lowest order the equation of motion (3.31) of the guiding centre has the form

$$\boldsymbol{F} = -q\dot{\boldsymbol{C}}\times\boldsymbol{B} + \mathrm{O}(\varepsilon), \qquad \boldsymbol{F}\cdot\boldsymbol{B} = \mathrm{O}(\varepsilon). \tag{3.34}$$

This is consistent with the assumptions made at the beginning of § 1. There it was stated that the longitudinal component of $\boldsymbol{F}$ should be small enough for large accelerations and velocities to be avoided along $\boldsymbol{B}$; otherwise condition (3.2) may be violated. We further observe that the first of expressions (3.34) requires the transverse force field $\boldsymbol{F}_\perp$ to be of the same order as the transverse drift velocity. Whether $\boldsymbol{F}_\perp$ is of zero or first order depends on the specific problem to be treated. If it is of zero order one should not forget that it will give important contributions to the inertia drift (3.25) as stated at the end of § 1.1.

Scalar multiplication of equations (3.32) and (3.33) by $\boldsymbol{B}$ and of (3.32) by $\boldsymbol{S}_1$ gives

$$\boldsymbol{C}_1\cdot\boldsymbol{B} = \mathrm{O}(\varepsilon), \qquad \boldsymbol{S}_1\cdot\boldsymbol{B} = \mathrm{O}(\varepsilon), \qquad \boldsymbol{C}_1\cdot\boldsymbol{S}_1 = \mathrm{O}(\varepsilon). \tag{3.35}$$

Here we do not permit $\dot{\vartheta}$ to be of first or higher order in ε since this would

vitiate the whole point of the representation (3.6). The result shows that the gyration can be described in first approximation by two vectors, $\boldsymbol{C}_1$ and $\boldsymbol{S}_1$, perpendicular to each other and to the magnetic field. Further, the difference between the scalar products of (3.32) with $\boldsymbol{C}_1$ and of (3.33) with $\boldsymbol{S}_1$ becomes after some deductions

$$C_1^2 = S_1^2 + \mathrm{O}(\varepsilon). \tag{3.36}$$

Thus, the vectors are equal in magnitude in the same approximation. Finally, combination of equations (3.32) and (3.33) yields

$$\dot{\vartheta} = B + \mathrm{O}(\varepsilon) \tag{3.37}$$

as expected from our basic assumptions. In fact the higher order terms of $\dot{\vartheta}$ can be chosen freely; this only adjusts the higher order contributions to $\boldsymbol{C}_\nu$ and $\boldsymbol{S}_\nu$ in (3.6) correspondingly. We shall return to this question in Ch. 4, § 3.

As a consequence of the obtained results we can represent the gyration of the particle by

$$\boldsymbol{a} = \varepsilon \boldsymbol{C}_1 \cos \omega_g t + \varepsilon \boldsymbol{S}_1 \sin \omega_g t + \mathrm{O}(\varepsilon) \tag{3.38}$$

in the first approximation where

$$\varepsilon \boldsymbol{C}_1 = \hat{\boldsymbol{B}} \times \boldsymbol{W}_0/\omega_g \,, \quad \varepsilon \boldsymbol{S}_1 = \boldsymbol{W}_0/\omega_g \tag{3.39}$$

in analogy with the solution (2.80) for the motion in a homogeneous magnetic field. This justifies the assumptions made in the first order theory of § 1.1.

In (3.31) we have obtained an equation of motion for the guiding centre up to second order. For the special case where the spatial variations of the force field $\boldsymbol{F}$ are much steeper than the corresponding variations of the magnetic field $\boldsymbol{B}$ we may still have to keep the second order contributions from $\boldsymbol{F}$ for a while, but neglect those originating from $\boldsymbol{B}$. This is the case, e.g., when $\boldsymbol{F} = q\boldsymbol{E}$ is the force from an electric field which is generated in a plasma over spatial dimensions that are much smaller than those of the immersed magnetic field. By the aid of equations (3.38) and (3.39) the result then becomes

$$m \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = (1 + \tfrac{1}{4}a^2 \nabla_\perp^2)\boldsymbol{F} + q\boldsymbol{u} \times \boldsymbol{B} - M\nabla B + \mathrm{O}(\varepsilon^2 \boldsymbol{B}) + \mathrm{O}(\varepsilon^3), \tag{3.40}$$

where $M = mW_0{}^2/2B$ is the equivalent magnetic moment and $\mathrm{O}(\varepsilon^2\boldsymbol{B})$ indicates terms of second order which involve derivatives of $\boldsymbol{B}$. The contribution to the external force from $a^2\nabla_\perp^2$ in equation (3.40) is due to the fact that the gyrating particle "feels" a mean force in the inhomogeneous field $\boldsymbol{F}$ which

is slightly different from the force $\boldsymbol{F}_C$ at the centre of gyration. This effect differs for ions and electrons when the corresponding radii of gyration a_i and a_e are unequal (ROSENBLUTH *et al.* [1962]). In a two-dimensional case $\nabla_\perp^2 \boldsymbol{E} = -\nabla(\nabla_\perp^2 \phi)$ differs from zero only when electric space charges are present.

In concluding this paragraph it should be stressed that one has to be careful in making approximations where terms of a certain order in the parameter ε are discarded. Important charge separation phenomena may arise from higher order terms only, and not from those of lowest order. This applies both to the equations of motion of the guiding centre and to those of ionized matter which will be deduced in the next paragraph and in Chapter 5. The relative importance of different terms in these equations is determined not only by their order in ε but also by their mathematical form and by the physical effects which they represent. An obvious example of this is the polarization drift (3.27) which gives a contribution of order ε in (3.22) as compared to the electric drift (3.26). The former produces a charge separation, but not the latter. An omission of the former is justified as long as only the first order orbit is studied for a particle in vacuo. However, the same approximation is not applicable to the dynamics of a dense plasma where charge separation is of vital importance. Another example is given by the second order term associated with $\boldsymbol{F}$ in (3.40) which has different values for ions and electrons and may give rise to important charge separation effects under certain circumstances. Finally, two examples will be demonstrated in Ch. 8, §§ 2.4 and 2.5 where higher order terms enter the equations of motion in such a way that they have a considerable influence on the final results.

2. Particle Flux and Currents

The guiding centre motion is only part of the total motion of a charged particle. When the flux of matter, momentum and energy is calculated account has to be taken not only of the flux of guiding centra but also of the contributions from the gyration of the particles. This is essential when the motion has to be linked to the electromagnetic and mechanical forces which act on an element of ionized matter. Starting from the single particle picture we shall now make a first order deduction of the particle current through a surface element.

The total flux of particles of density n through a surface of area S is

$$\Psi = \iint_S n\hat{\boldsymbol{n}} \cdot \boldsymbol{w}\,\mathrm{d}S = \iint_S n\hat{\boldsymbol{n}} \cdot (\boldsymbol{u} + \boldsymbol{W})\mathrm{d}S \equiv \Psi_u + \Psi_W, \quad (3.41)$$

where Ψ_u and Ψ_W represent the contributions from the guiding centre drift and the gyration, and $\boldsymbol{u}$ and $\boldsymbol{W}$ are defined in equations (3.9) and (3.6). The situation is shown in Figure 3.4, where we observe that only particles which penetrate the area S once give contributions to Ψ_W (cf. CHANDRASEKHAR and TREHAN [1960]). Such particles are situated in a channel-like volume which follows the boundary C of the surface S. Every particle inside this volume cuts the surface $\omega_g/2\pi$ times per second. As volume elements we choose small discs with area πa^2 and thickness $\hat{\boldsymbol{B}} \cdot \mathrm{d}\boldsymbol{l}$. Here a is the radius of gyration, which is assumed to be the same for all particles at a particular point, and $\mathrm{d}\boldsymbol{l}$ is a line element of C. These volume elements contain all particles penetrating S once. With the positive directions defined in Figure

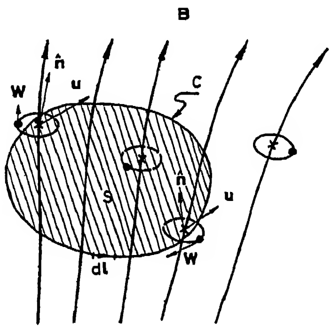


Fig. 3.4. The total flux of particles through a surface S is the sum of the guiding centre flux $n\boldsymbol{u}$ and a contribution from the gyration velocity $\boldsymbol{W}$ (cf. CHANDRASEKHAR and TREHAN [1960]).

3.4. and with the equivalent magnetic moment $\boldsymbol{M}$ defined by (2.83) to lowest order we obtain

$$\Psi_W = - \oint_C (n\omega_g/2\pi)\pi a^2 \hat{\boldsymbol{B}} \cdot \mathrm{d}\boldsymbol{l}$$

$$= \frac{1}{q} \oint_C n\boldsymbol{M} \cdot \mathrm{d}\boldsymbol{l} = \iint_S \hat{\boldsymbol{n}} \cdot \mathrm{curl}\left(\frac{n\boldsymbol{M}}{q}\right) \mathrm{d}S \tag{3.42}$$

in first order. Consequently, we can define a *macroscopic velocity* (mass velocity) $\boldsymbol{v}$ which represents the mean flux of particles per unit area and is determined by

$$n\boldsymbol{v} = n\boldsymbol{u} + \mathrm{curl}\,(n\boldsymbol{M}/q) \tag{3.43}$$

in the same order. The left hand member of equation (3.43) gives the total particle current density. It is the sum of the densities of the total drift current

and the magnetization current given by the first and second terms of the right hand member, respectively.

In reality there is a continuous distribution of the values of a^2 as well as of the particles in velocity space. Then, the contributions of a^2 to the flux at a certain point in space should be integrated over the velocity distribution. For any interval of $a^2 = W^2/\omega_g{}^2$ between a^2 and $a^2 + d(a^2)$ we can repeat the deductions just demonstrated. We arrive at relations of the forms (3.42), (3.43) and (3.24) where M, W^2 and $u_{\parallel}{}^2$ should be replaced by their square mean values $\overline{\boldsymbol{M}}$, $\overline{(W^2)}$ and $\overline{(u_{\parallel}{}^2)}$ in velocity space.

In this connexion it has to be pointed out again that W^2 and $u_{\parallel}{}^2$ are not constants of the motion of a particle. Therefore $\overline{W^2}$ and $\overline{u_{\parallel}{}^2}$ may change in space and time on account of the particle motion.

For the flux density $n\boldsymbol{v}$ of a certain kind of particles we can now write down an explicit expression by the help of equations (3.43) and (3.22). From some well-known vector identities

$$\begin{aligned} n\boldsymbol{v} &= n\overline{\boldsymbol{u}}_{\parallel} + n\overline{\boldsymbol{u}}_{\perp} - \operatorname{curl} \overline{(nmW^2\boldsymbol{B}/2qB^2)} \qquad (3.44) \\ &= n\overline{\boldsymbol{u}}_{\parallel} + n\boldsymbol{F} \times \boldsymbol{B}/qB^2 + nm\overline{(W^2 - 2u_{\parallel}^2)}\nabla B \times \boldsymbol{B}/2qB^3 \\ &\quad + nm\overline{(u_{\parallel}^2 - \tfrac{1}{2}W^2)}\,(\operatorname{curl}\boldsymbol{B})_{\perp}/qB^2 - nm\left(\frac{\mathrm{d}\boldsymbol{u}_{\perp}}{\mathrm{d}t}\right) \times \boldsymbol{B}/qB^2 \\ &\quad - \tfrac{1}{2}nm\overline{W^2}(\operatorname{curl}\boldsymbol{B})_{\parallel}/qB^2 - \nabla(nm\overline{W^2}) \times \boldsymbol{B}/2qB^2. \end{aligned}$$

Here we can write $\boldsymbol{u}_{\parallel} = \overline{\boldsymbol{u}}_{\parallel} + \tilde{\boldsymbol{u}}_{\parallel}$, where $\overline{\boldsymbol{u}}_{\parallel}$ is the mean longitudinal drift of the particles at a particular point in space and $\tilde{\boldsymbol{u}}_{\parallel}$ is the "thermal" part of the longitudinal drift velocity. For a symmetric distribution of $\tilde{\boldsymbol{u}}_{\parallel}$ around $\tilde{\boldsymbol{u}}_{\parallel} = 0$ in velocity space (such as a Maxwellian distribution) we have $\overline{(\overline{\boldsymbol{u}}_{\parallel} \cdot \tilde{\boldsymbol{u}}_{\parallel})} = 0$. Introduce the mean kinetic energies per particle

$$K_{\parallel} = \tfrac{1}{2}m\overline{u_{\parallel}^2} = \tfrac{1}{2}m(\overline{u}_{\parallel}^2 + \overline{\tilde{u}_{\parallel}^2}), \quad K_{\perp} = \tfrac{1}{2}m\overline{W^2} \qquad (3.45)$$

for the longitudinal motion and the gyration, respectively. Since the thermal motion is usually much more rapid than the mean motion we have $\overline{\boldsymbol{u}}_{\parallel}{}^2 \ll \overline{\tilde{\boldsymbol{u}}_{\parallel}{}^2}$ and $K_{\parallel}$ and $K_{\perp}$ will represent the longitudinal and transverse "temperatures" of the particles.

It should be pointed out that a definition of local values of the mean kinetic energies (3.45) makes only sense when the longitudinal and transverse drifts of the particles are slow compared to the rate at which the velocity distribution changes. Thus, to be able to define mean values (3.45) we have to

assume that a coupling due to collisions is operative between different parts of the velocity spectrum. We can still assume the system to be approximately dissipation-free. To take a specific example, this statement means that we could assume the particles to have velocity distributions which are nearly Maxwellian due to the effect of collisions, and still neglect Ohmic and viscous dissipation.

With the present notations equation (3.44) can be written as

$$n\boldsymbol{v}_{\|} = n\overline{\boldsymbol{u}}_{\|} - nK_{\perp}(\text{curl}\,\boldsymbol{B})_{\|}/qB^2 \tag{3.46}$$

and

$$n\boldsymbol{v}_{\perp} = n\boldsymbol{F} \times \boldsymbol{B}/qB^2 + n(K_{\perp} - 2K_{\|})(\nabla B - \hat{\boldsymbol{B}} \times \text{curl}\,\boldsymbol{B}) \times \boldsymbol{B}/qB^3$$
$$- \nabla(nK_{\perp}) \times \boldsymbol{B}/qB^2 - nm\,\overline{\frac{\mathrm{d}\boldsymbol{u}_{\perp}}{\mathrm{d}t}} \times \boldsymbol{B}/qB^2. \tag{3.47}$$

These equations have been deduced from a *first order* theory and should be used with care when higher order approximations are concerned. This applies especially to the inertia terms. For a more accurate evaluation of the total particle flux we have to use higher order expressions, not only for the contribution from the guiding centra, but also for that originating from the Larmor motion. Further considerations of this problem will be made in Ch. 5, § 2.

Observe here that the longitudinal flux of particles is not only given by the contribution $n\overline{\boldsymbol{u}}_{\|}$ from the guiding centre drift in equation (3.46). There is also a contribution from the gyration which is connected with the curvature of the magnetic field lines. Further, the contributions from the derivatives of the magnetic field to the transverse flux of (3.47) will vanish when the velocity distribution is isotropic and $K_{\perp} = 2K_{\|}$.

Up to this point we have not discussed the balance of forces in the longitudinal direction. For this purpose consider a volume element of thickness $\mathrm{d}s$ which is bounded by two surfaces of areas S and $S + \mathrm{d}S$ as shown in Figure 3.5. The surfaces are perpendicular to the magnetic field lines and enclose the same magnetic flux $\Phi = B \cdot S$. Particles inside the element are subject to a volume force $n\boldsymbol{F} \cdot \hat{\boldsymbol{B}} - n\overline{M}(\hat{\boldsymbol{B}} \cdot \nabla)B$ in the longitudinal direction according to equation (3.17). In addition, there is a pressure force $2nK_{\|}S = p_{\|}S$ on each of the bounding surfaces in the longitudinal direction. The resulting total force is balanced by the longitudinal inertia force of the total

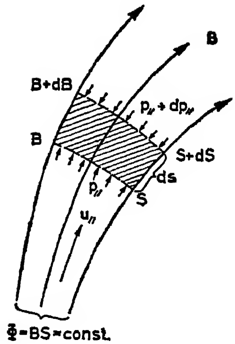


Fig. 3.5. Volume element of thickness ds which is bounded by two surfaces of areas S and $S + \mathrm{d}S$. The latter are perpendicular to the magnetic field.

mass flow. Thus,

$$nm\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right]_{\parallel} S\mathrm{d}s$$
$$= n[\boldsymbol{F}\cdot\hat{\boldsymbol{B}} - \overline{M}(\hat{\boldsymbol{B}}\cdot\nabla)B]S\mathrm{d}s - \frac{\partial}{\partial s}(2nK_{\parallel}S)\mathrm{d}s. \qquad (3.48)$$

Using the definition of M and the relation $S = \Phi/B = \text{const.}/B$ we obtain

$$nm\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right]_{\parallel} = n\boldsymbol{F}_{\parallel} - (n/B)(K_{\perp} - 2K_{\parallel})\nabla_{\parallel}B - \nabla_{\parallel}(2nK_{\parallel}). \qquad (3.49)$$

as the equation of motion of the mass flow in the longitudinal direction. The connexion of equations (3.47) and (3.49) with the macroscopic theory will be discussed more in detail in Ch. 5, § 2.

In closing this paragraph we shall consider the special case were the kinetic energies $K_{\parallel}$ and $K_{\perp}$ are small enough for the electric field drift $\boldsymbol{u}_E$ to provide the only inertia effect of importance. We also linearize the problem by putting $\mathrm{d}\boldsymbol{u}_E/\mathrm{d}t = \partial\boldsymbol{u}_E/\partial t$. For a gas mixture of ions and electrons the electric current density then becomes

$$\boldsymbol{j} = \boldsymbol{j}_{\parallel} + \boldsymbol{j}_{\perp} \approx en_{\mathrm{i}}(\bar{\boldsymbol{u}}_{\parallel\mathrm{i}} + \boldsymbol{u}_E) - en_{\mathrm{e}}(\bar{\boldsymbol{u}}_{\parallel\mathrm{e}} + \boldsymbol{u}_E) + \varepsilon_{\mathrm{eq}}\frac{\partial \boldsymbol{E}_{\perp}}{\partial t}, \qquad (3.50)$$

where

$$\varepsilon_{\mathrm{eq}} = (n_i m_i + n_e m_e)/B^2. \qquad (3.51)$$

If expression (3.50) for the current density is substituted into equation (2.2) we see that the gas behaves like an anisotropic dielectric medium with the

dielectric constants ε_0 and $\varepsilon_0 + \varepsilon_{eq}$ in the longitudinal and transverse directions, respectively. This result has been deduced under the assumption of a small thermal energy and is by no means of general validity. Thus, the behaviour of a plasma cannot always be described simply by a substitution of ε_0 by $\varepsilon_0 + \varepsilon_{eq}$ in the transverse part of (2.2). Instead we have to investigate every particular problem in detail before such a substitution can be justified. This should be done with special attention to charge separation phenomena.

In special cases, such as when plane elementary waves are to be studied and there is no need to write down equations for plasma motions of arbitrary form, it is of course possible to connect $\boldsymbol{E}$ and $\boldsymbol{B}$ by an equation analogous to (2.2). A dielectric tensor then replaces ε_0, but it becomes a function of the frequency ω of the wave at the same time (cf. DELCROIX [1960]). Since the tensor depends on the properties of the particular wave motion, it is not a simple constant like (3.51) which is determined by the mass density and the strength B only.

* 3. Hamiltonian Formulation of the Guiding Centre Approach

So far we have applied the perturbation theory directly to the equation of motion of the particle. Alternatively, the Hamiltonian formalism of Ch. 2, § 3 can be used as a starting point in a development where the particle motion is divided essentially into a gyration around the field lines superimposed on a guiding centre drift. Approaches of this kind are due to GARDNER [1959] and TANIUTI [1961].

Following Gardner we study a charged particle moving in an electric field $\boldsymbol{E}$, and in a magnetic field $\boldsymbol{B}$ which has no zero point in the actual regions of space. We assume the longitudinal component $\boldsymbol{E} \cdot \hat{\boldsymbol{B}}$ to be of order ε. Here ε is the dimensionless "smallness parameter" defined in § 1. It represents the ratio between the Larmor radius and the characteristic dimensions of the electromagnetic field as well as the ratio between the gyro period and the characteristic times of the same fields. Thus, we impose the same restrictions on the motion as those given by conditions (3.1) and (3.2) of the orbit theory. Since both $\boldsymbol{E}$ and $\boldsymbol{B}$ should vary slowly in space and time we write the potentials $\boldsymbol{A}$ and ϕ of equations (2.8), (2.10) and (2.21) in the following manner:

$$q\boldsymbol{A}(\boldsymbol{\rho}, t) = \left(\frac{q}{\varepsilon}\right)\boldsymbol{A}'(\varepsilon\boldsymbol{\rho}, \varepsilon t) = \frac{1}{\varepsilon}\alpha\nabla\beta, \tag{3.52}$$

$$\phi(\boldsymbol{\rho}, t) = \frac{1}{\varepsilon}\phi'(\varepsilon\boldsymbol{\rho}, \varepsilon t). \tag{3.53}$$

Here we have used the gauge with $\nabla\chi = 0$ in equation (2.21). As compared to Ch. 2, § 1.2 the definitions are now slightly modified in the way that a factor q is included in $\alpha\nabla\beta$ and the scalar quantities α and β are understood to be functions of $\varepsilon\boldsymbol{\rho}$ and εt. Since ε is small compared to unity the slow relative variation of $\boldsymbol{A}$ and ϕ, and of $\boldsymbol{E}$ and $\boldsymbol{B}$, in space and time is expressed directly by means of the forms (3.52) and (3.53). A gravitation field arising from the potential ϕ_g can easily be included here, but will be left out for the sake of simplicity.

When the generalized coordinates are rectangular coordinates x_k and the corresponding momenta p_k are determined by (2.61) the velocity $\boldsymbol{w}$ of the particle is given by

$$m\boldsymbol{w} = \boldsymbol{p} - (q/\varepsilon)\boldsymbol{A}' \tag{3.54}$$

and the Hamiltonian (2.62) becomes

$$H = \frac{1}{2m}\left(\boldsymbol{p} - \frac{q}{\varepsilon}\boldsymbol{A}'\right)^2 + \frac{q}{\varepsilon}\phi'. \tag{3.55}$$

We shall now change the representation to a new set of generalized coordinates and momenta $(q_1, q_2, q_3; p_1, p_2, p_3)$ by means of a canonical transformation. In this new representation the motion should be split essentially into three parts corresponding to the gyration velocity $\boldsymbol{W}$ and the longitudinal and transverse drifts $\boldsymbol{u}_\parallel$ and $\boldsymbol{u}_\perp$ of the first order orbit theory. For this purpose we try to use a generating function of the form discussed in Ch. 2, § 3.1.:

$$G(x, y, z; p_1, p_2, p_3; t) = \frac{s}{\varepsilon}p_2 + \frac{\beta}{\varepsilon}p_1 + \frac{\alpha}{\varepsilon}p_3 - p_1 p_3, \tag{3.56}$$

where $s(\varepsilon\rho, \varepsilon t)$ corresponds to the arc length measured along a magnetic field line. Conditions (2.58) for the canonical transformation immediately yield

$$\alpha = \varepsilon p_1 + \varepsilon q_3, \quad \beta = \varepsilon q_1 + \varepsilon p_3, \quad s = \varepsilon q_2, \tag{3.57}$$

$$\boldsymbol{p} - \frac{q}{\varepsilon}\boldsymbol{A}' = \frac{1}{\varepsilon}(\nabla s)p_2 + \frac{1}{\varepsilon}(\nabla\alpha)p_3 - \frac{1}{\varepsilon}(\nabla\beta)q_3, \tag{3.58}$$

and a new Hamiltonian

$$H' = \frac{1}{\varepsilon}\left[q\phi' + \frac{1}{\varepsilon}\left(\frac{\partial\beta}{\partial t}\right)(\varepsilon p_1)\right] \tag{3.59}$$

$$+ \frac{1}{2m}\left[\frac{1}{\varepsilon}(\nabla s)p_2 + \frac{1}{\varepsilon}(\nabla\alpha)p_3 - \frac{1}{\varepsilon}(\nabla\beta)q_3\right]^2 + \frac{1}{\varepsilon}\left(\frac{\partial s}{\partial t}\right)p_2 + \frac{1}{\varepsilon}\left(\frac{\partial\alpha}{\partial t}\right)p_3,$$

where use has been made of equations (3.57), (3.58) and (3.52). When εp_1 in the first of relations (3.57) is substituted into (3.58) we obtain

$$H' = \frac{1}{\varepsilon}\left[q\phi' + \frac{\alpha}{\varepsilon}\left(\frac{\partial \beta}{\partial t}\right)\right] + \tfrac{1}{2}mw^2 + \frac{1}{\varepsilon}\left(\frac{\partial s}{\partial t}\right)p_2 + \frac{1}{\varepsilon}\left(\frac{\partial \alpha}{\partial t}\right)p_3 - \frac{1}{\varepsilon}\left(\frac{\partial \beta}{\partial t}\right)q_3. \tag{3.60}$$

Observe that in the new representation the particle motion has three degrees of freedom which now correspond to (q_1, p_1), (q_2, p_2) and (q_3, p_3).

According to equations (3.58) and (3.54) p_2, p_3, q_3 are essentially components of velocity and should therefore be of order unity, since $(1/\varepsilon)(\partial/\partial x_k) = \partial/\partial(\varepsilon x_k)$ and s, α, β are essentially geometric coordinates. Then equation (3.57) shows that εp_1, εq_1, εq_2 are essentially geometric coordinates of order unity and εq_3, εp_3 are small deviations, of order ε, from the positions α, β in planes perpendicular to $\boldsymbol{B}$. In first order (q_3, p_3) therefore represent the velocity $\boldsymbol{W}$ of gyration. The variables $(\varepsilon q_2, p_2)$ correspond to the coordinate s and the velocity $\boldsymbol{u}_{\parallel}$ of the guiding centre along $\boldsymbol{B}$. Finally $(\varepsilon q_1, \varepsilon p_1)$ represent the position of the guiding centre on the field lines and its drift across $\boldsymbol{B}$. Only in a first approximation can the three motions $\boldsymbol{W}$, $\boldsymbol{u}_{\parallel}$ and $\boldsymbol{u}_{\perp}$ be associated with three degrees of freedom; in higher approximations there will be a coupling between them. Such a coupling is also connected with deviations from the adiabatic invariance discussed in the next chapter.

The obtained form (3.60) of the Hamiltonian can be made plausible by means of some simple arguments. Since α, β, s are defined for a time-dependent magnetic field, the field lines act as moving constraints and the particle will "slide" on them like a bead on a moving wire. Then, the Hamiltonian will not be connected with the total potential energy from the work of the electric field $\boldsymbol{E}$. It will only contain the work of such forces which are considered as "external" or "applied" forces in the new representation. Thus, we have to exclude the part of the work which is now considered to originate from forces of constraint, i.e., from the motion of the field lines.

The work of the forces of constraint is of two types. Firstly, their transverse component gives a contribution when there is a displacement of the field lines in the transverse direction, i.e. when $(\partial\alpha/\partial t)$ and $(\partial\beta/\partial t)$ differ from zero. This produces a work by the transverse force of constraint, like that performed on a bead which slides on a moving curve of constant shape. Secondly, a contribution arises when the length of the field lines varies and $(\partial s/\partial t)$ differs from zero. This produces a work similar to that which is generated when a bead slides on a ring-shaped elastic wire, the radius and arc length of which change in time.

To connect these physical pictures with (3.60) we now consider the total work $q\boldsymbol{E} \cdot d\boldsymbol{l}$ performed by the electric field, as given in equation (2.23). One of the terms in the expression for this work is $(1/\varepsilon^2)(\partial\beta/\partial t)(\nabla\alpha) \cdot d\boldsymbol{l} - (1/\varepsilon^2)(\partial\alpha/\partial t)(\nabla\beta) \cdot d\boldsymbol{l}$ with the present notation. The gradients of α and β can each be divided into two contributions according to equation (3.57). One is due to (q_3, p_3) and the other to (q_1, p_1). The latter comes mainly from the drift of particles across the magnetic field and is therefore responsible for the transverse work of the forces of constraint. In addition, the longitudinal work of the same forces should be of the form $(1/\varepsilon)(-\partial s/\partial t)p_2$, since $(1/\varepsilon)(\partial s/\partial t)$ represents the rate of change of the arc length along $\boldsymbol{B}$ and p_2/m the longitudinal velocity. Thus, from the energy corresponding to the integral of $-\boldsymbol{E} \cdot d\boldsymbol{l}$ we subtract the transverse and longitudinal works of the forces of constraint, and add the kinetic energy $\frac{1}{2}mw^2$ to get the Hamiltonian. This results in an expression of the form (3.60), where the last two terms represent the part of $(\partial\beta/\partial t)\nabla\alpha - (\partial\alpha/\partial t)\nabla\beta$ which remains when this subtraction has been made.

The coefficients in (3.60) are evaluated in terms of $(q_1, q_2, q_3; p_1, p_2, p_3)$ by use of relations (3.57) and by setting $\alpha = \varepsilon p_1$, $\beta = \varepsilon q_1$ and $s = \varepsilon q_2$. We now expand H' in powers of q_3, p_3 since the terms εq_3, εp_3 in equations (3.57) are of order ε and εq_1, εp_1 are of order unity. Then, the Hamiltonian has the form

$$H' = \frac{1}{\varepsilon} H_{-1} + H_0 + \varepsilon H_1 + \cdots \tag{3.61}$$

The first terms in this series are

$$H_{-1} = q\phi' + \frac{\alpha}{\varepsilon}\left(\frac{\partial\beta}{\partial t}\right) \tag{3.62}$$

and

$$H_0 = \tfrac{1}{2}m\left[W_3^2 + \left(\frac{p_2}{m}\right)^2 + \frac{2p_2}{m\varepsilon}\left(\frac{\partial s}{\partial t} + \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}\cdot\nabla s\right) - \frac{E^2}{B^2}\right]. \tag{3.63}$$

Here we have defined

$$\boldsymbol{W}_3 = \boldsymbol{w} - (p_2/m)\hat{\boldsymbol{B}} - \boldsymbol{E}\times\boldsymbol{B}/B^2. \tag{3.64}$$

In first order this expression becomes equal to the velocity of gyration $\boldsymbol{W}$, since the second term of the right hand member corresponds to $\boldsymbol{u}_{\parallel}$ and the last term approaches $\boldsymbol{u}_{\perp}$ according to equation (3.34).

Especially for a static electromagnetic field treated by TANIUTI [1961] all

terms of (3.60) vanish which have an explicit time dependence. In a first order theory we can define the velocities according to (3.9) and the magnetic moment $M = mW^2/2B$ is given by equation (2.83). The transformed Hamiltonian then reduces to

$$H' = q\phi + MB + \tfrac{1}{2}mu^2. \tag{3.65}$$

As we shall see in the next chapter the magnetic moment is an approximate constant of the motion in first order. This implies that the explicit dependence of the gyration has disappeared in (3.65) which has effectively been reduced to only the mean motion of the guiding centre. The term MB behaves here like an additional potential energy. Among other things, this equivalent potential will appear in the magnetic mirror effect acting on the gyrating particle as discussed in Ch. 2, § 4.2(i), Ch. 6, § 2.1 and Ch. 7, § 3.1. The present result (3.65) is also what should be expected from the equation of motion (3.16) of the guiding centre in a slowly varying electromagnetic field. The forms of equations (2.36) and (3.16) differ only by the "magnetic gradient force", $-M\nabla B$.

In Ch. 4, § 1 we shall see that canonical equations can sometimes be defined also for the mean motion of particles which oscillate between two magnetic mirrors and drift around a field configuration in nearly closed orbits.

CHAPTER 4

ADIABATIC INVARIANTS

The orbit theory of Chapter 3 shows that the complicated motion of a charged particle often can be described quite accurately by a superposition of a drift motion and a gyration. Both these motions change slowly during a Larmor period. In fact, the behaviour of the particle is characterized by a number of quantities which change slowly enough to become approximate constants of the motion. They are not exact integrals of the latter, but will approach a constant value in the *limit of infinitely slow* variations of the external parameters. Quantities which behave in this manner are denoted as *adiabatic invariants*. As we shall see here and in Chapter 5, § 2.2 these invariants are also directly connected with the adiabatic compression of ionized matter in a magnetic field.

A unification of the investigations made so far on adiabatic invariants has recently been presented by KRUSKAL [1962].

1. First Order Theory

That the *equivalent magnetic moment* of a gyrating particle is an approximate constant of the motion was first realized by ALFVÉN [1940, 1950]. This property greatly simplifies the analytical treatment of particle orbits in terms of the perturbation theory of Ch. 3, § 1. Alfvén's investigations were inspired by studies of the aurora and of cosmic radiation. In later investigations on the acceleration of cosmic radiation FERMI [1954] proposed a mechanism where particles move along the field lines and "bounce" between two magnetic mirrors which approach each other. This mechanism is connected with the *longitudinal invariant* suggested by ROSENBLUTH [1956] and further developed by NORTHROP and TELLER [1960], as demonstrated in § 1.3 of the present chapter. In a study of the Van Allen belts in the earth's magnetic field the latter authors also have defined a *flux invariant*. This invariant is associated with the magnetic flux enclosed by the orbit of the guiding centre motion. It is closely related to the conditions for flux-preservation and for ionized matter to be "frozen" in a strong magnetic field.

1.1. GENERAL CONSIDERATIONS

Before a detailed analysis is undertaken we shall make some simple considerations on the three invariants just mentioned. According to equations (3.6) and (3.9) the particle motion consists in first order of a gyration at the velocity $\boldsymbol{W}$ and a drift motion with longitudinal and transverse components $\boldsymbol{u}_{\parallel}$ and $\boldsymbol{u}_{\perp}$. The position vectors of the particle and the centre of gyration are $\boldsymbol{\rho}$ and $\boldsymbol{C}$, and $\boldsymbol{a} = \boldsymbol{\rho} - \boldsymbol{C}$ is a vector connected with the Larmor radius.

There are at least three periodic motions which the particle may perform simultaneously in a field such as that sketched in Figure 4.1. Firstly, it will

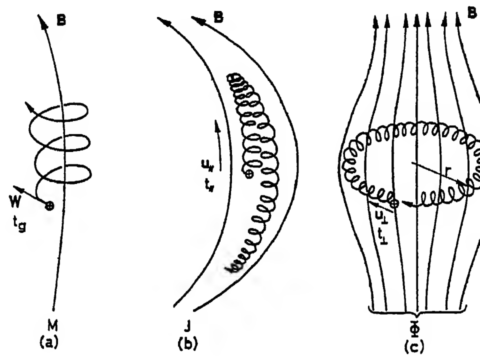


Fig. 4.1. Periodic motions of a charged particle in a magnetic field. a. Gyration at velocity $\boldsymbol{W}$ around the field lines and with the gyro period t_g. b. Oscillations between magnetic mirrors at velocity $\boldsymbol{u}_{\parallel}$ along the field lines and with the period $t_{\parallel}$. c. Repeated drift around the whole configuration at velocity $\boldsymbol{u}_{\perp}$ across the field lines and with the period $t_{\perp}$

gyrate around the field lines at the velocity $\boldsymbol{W}$ and with the gyro period $t_g = 2\pi/|\omega_g|$ as shown in Figure 4.1a. Secondly, it may oscillate between two magnetic mirrors at the velocity $\boldsymbol{u}_{\parallel}$ and with the period $t_{\parallel}$ (Figure 4.1b). Finally, in a magnetic field of suitable form the transverse drift orbit may also become closed and the particle revolves around the configuration at the drift velocity $\boldsymbol{u}_{\perp}$ with a period $t_{\perp}$ (Figure 4.1c). Usually $t_g \ll t_{\parallel} \ll t_{\perp}$.

It is easily seen from some crude physical arguments that there are three invariants associated with these motions. The situation in Figure 4.1a is analogous to that of a mass point which is attached to a wire. It revolves around an axis at a distance equal to the radius, a, of gyration. The stress in the wire corresponds to the force provided by the magnetic field. Assume the system to be dissipation-free and the radius a to change slowly. Then, conservation of angular momentum requires $mWa = mW^2t_g/2\pi$ to be con-

stant. Consequently, equations (2.81) and (2.83) suggest that the equivalent magnetic moment M should become a constant as well.

This is also obvious from the thermodynamic point of view. Thus, consider W to represent a thermal motion corresponding to a temperature $T = mW^2/2k$ in a gas of particle density n. In absence of dissipation a slow change of the radius a then corresponds to a two-dimensional adiabatic compression or expansion, where the density n is proportional to both $1/a^2$ and to T. This implies that W^2/B and M should be constant.

We can finally assume the revolving particle to be substituted by a ring current in a flexible superconductor which encloses the magnetic flux $\pi a^2 B$. When the radius a and the magnetic field B change slowly in time, the induced electric currents can only remain finite if the electromotive force vanishes and $a^2 B$ and M become invariants. A first order proof and a simple physical discussion on the constancy of M will be given in § 1.2 of this chapter.

Further, when the transverse drift from field line to field line is slow compared to the longitudinal oscillations, the situation of Figure 4.1b resembles that of an elastic ball which bounces between two moving pistons. The latter represent the magnetic mirrors. Their motion arises from the variations of the magnetic field in space and time, as experienced by the drifting and oscillating particle. As will become obvious from the simple analysis of Figure 4.4 at the beginning of § 1.3, the momentum balance then requires $u_{\|}^2 t_{\|}$ to be constant. This is also to be expected from the thermodynamic point of view. Thus, we consider a one-dimensional compression or expansion of a gas in the directions along the magnetic field lines. With a temperature $T = mu_{\|}^2/k$ adiabatic changes of state require the density to become proportional to both $(u_{\|} t_{\|})^{-1}$ and to $T^{\frac{1}{2}}$. This shows that $u_{\|}^2 t_{\|} = u_{\|}(u_{\|} t_{\|})$ should be invariant, where $u_{\|} t_{\|}$ is double the distance between the magnetic mirrors of Figure 4.4.

Finally, for the motion sketched in Figure 4.1c we introduce a cylindrical coordinate system (r, φ, z) with z along the axis of symmetry. To simplify the discussion we assume the particle to have no velocity along the field lines and to move in the "equatorial" plane of the figure where the magnetic field lines are all parallel with the z axis. In absence of external force fields and with $\boldsymbol{B} = \operatorname{curl} \boldsymbol{A}$, the radial drift velocity then becomes $u_r = -(1/B)(\partial A_\varphi/\partial t)$ to lowest order. When the magnetic field changes in time, the surfaces of constant enclosed flux, $\Phi = 2\pi r A_\varphi = \text{const.}$, move in the radial direction in the equatorial plane. It is then immediately seen that $(\partial/\partial t + u_r \partial/\partial r)\Phi$ vanishes. This implies that the flux Φ enclosed by the particle orbit becomes

invariant for sufficiently slow variations of the field. The orbit of the drift motion then behaves like a flexible ring-shaped superconductor which changes its radius in such a way that the induced electromotive force around its perimeter vanishes. In other words, the particle drift becomes flux-preserving in the sense of Ch. 2, § 1.3.

We shall now make a more detailed analysis of the situation in terms of the action integrals of Ch. 2, § 3.3. The total motion of the particle of Figures 4.1a, b and c has *three* degrees of freedom and is represented essentially by the velocity $\boldsymbol{W}$ of gyration and the longitudinal and transverse drift velocities $\boldsymbol{u}_{\parallel}$ and $\boldsymbol{u}_{\perp}$. To lowest order these motions are also represented by the canonical coordinates (q_3, p_3), $(\varepsilon q_2, p_2)$ and $(\varepsilon q_1, \varepsilon p_1)$ of Ch. 3, § 3. We shall first make a mean value formation over the gyration. The result is a system having a Hamiltonian with *two* degrees of freedom which describes the longitudinal and transverse drifts of the guiding centre and where the magnetic moment M is included as a parameter. Secondly, we make a mean value formation over the longitudinal motion of this system. We then obtain a new system having a Hamiltonian with *one* degree of freedom. It describes the mean transverse drift of the guiding centre and contains both M and the longitudinal invariant J as parameters. We shall finally make a mean value formation over the mean transverse drift around a configuration like that given in Figure 4.1c. The magnetic flux Φ enclosed by the orbit of this drift will then turn out to be a constant of the motion.

We first consider the total velocity $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{W}$ of the particle in an electromagnetic field of arbitrary shape. The generalized momentum (2.61) can be written as

$$\boldsymbol{p} = m(\boldsymbol{u} + \boldsymbol{W}) + q\boldsymbol{A}. \tag{4.1}$$

The motion to be studied is "nearly periodic" and can be treated by a method introduced by KRUSKAL [1957, 1958] and described in Ch. 2, § 3.3. With respect to the gyration we consider the generalized momenta and space coordinates to have a periodic time dependence given by a function $\vartheta(t)$ as well as an explicit and slow time dependence which is not related to the periods of the system, as stated in Chapter 2, § 3.3. The function ϑ is connected with the frequency ω_g of gyration as indicated in equations (3.6) and (3.37). The assumption of a time dependence of the form (2.64) with $\Theta = \vartheta$ is justified since it has been shown by BERKOWITZ and GARDNER [1959] that the exact solution of the equation of motion can be expressed by the

expansion (3.6). Consequently, we write the action integral (2.71) as

$$J^* = \oint [m(\boldsymbol{W} + \boldsymbol{u}) + q\boldsymbol{A}]_t \frac{\partial}{\partial \vartheta} (\boldsymbol{C} + \boldsymbol{a})_t \, \mathrm{d}\vartheta = \text{const.}, \tag{4.2}$$

where $\boldsymbol{W} = \mathrm{d}\boldsymbol{a}/\mathrm{d}t$, $\boldsymbol{u} = \mathrm{d}\boldsymbol{C}/\mathrm{d}t$. Further, t should be kept constant in all places where it appears explicitly. Observe that $\boldsymbol{u}$ and $\boldsymbol{C}$ do not depend on ϑ.

We now make a first order estimation of the action integral (4.2). Since $\boldsymbol{u}$ and $\boldsymbol{A}$ vary very little during a period t_g of ϑ we write

$$J^* \approx \oint m \left[\boldsymbol{W} \cdot \frac{\partial \boldsymbol{a}}{\partial \vartheta} \right]_t \mathrm{d}\vartheta = m \oint \boldsymbol{W} \cdot \mathrm{d}\boldsymbol{a} = m \int_0^{t_g} W^2 \mathrm{d}t \tag{4.3}$$

because $\mathrm{d}\boldsymbol{a}/\mathrm{d}t = (\partial \boldsymbol{a}/\partial \vartheta)\mathrm{d}\vartheta$ when time is kept constant in places where it appears explicitly. Remembering that $\boldsymbol{W} = \mathrm{d}\boldsymbol{a}/\mathrm{d}t$ and that W^2 does not change very much during a Larmor period we obtain

$$J^* \approx mW^2 t_g \approx \text{const.} \tag{4.4}$$

The gyro period is $t_g = 2\pi m/|q|B$ and the equivalent magnetic moment is $M = mW^2/2B$. Hence

$$M \approx (|q|/4\pi m)J^* \approx (|q|/4\pi)W^2 t_g \approx \text{const.} \tag{4.5}$$

and M becomes an approximate constant of the motion.

In the present proof of the constancy of M we have made a direct study of the true orbit of the particle and of oscillations which originate from the Larmor motion. We are on less safe grounds when we try to develop an analogous treatment for the longitudinal oscillations of the mean motion sketched in Figure 4.1b. To be able to study a motion of this kind by means of an action integral of the form (2.71) we have to assume that Θ is now connected with the frequency $\vartheta_{\parallel} = 2\pi/t_{\parallel}$ of a periodic longitudinal motion. Contrary to our previous study of the Larmor motion there is no expansion available which includes $\vartheta_{\parallel}$ in a form analogous to (3.6) for ϑ. We will only assume, as ROSENBLUTH [1956] has done, that a tightly bound gyrating particle moves back and forth along a magnetic field line and returns *nearly* to its initial state after one "period" $t_{\parallel}$. In other words, we assume that the Larmor radius and the period of gyration are very small compared to the changes of the electromagnetic field in space and time. This is also connected with the requirement that the particle should drift quite a small distance across the field during a longitudinal period $t_{\parallel}$. The spread of two consecutive turning points will then be of the order of one Larmor radius. As

pointed out by NORTHROP [1961] this requires the transverse component $\boldsymbol{F}_\perp$ of the force field to be of first order in ε. If it is of zero order, the drift (3.23) would namely displace the particle too far across the field lines during a period $t_\parallel$, and our assumptions about the periodicity of the longitudinal motion are no longer relevant.

It further has to be observed that it was possible to use the complete action integral J^* of (2.71) in the previous discussion on the magnetic moment. Such a treatment is not possible in case of the longitudinal invariant, since the latter is associated with only one of the terms in the series (2.71).

For a closer examination of these problems consider the equation of the longitudinal drift motion which is obtained from equation (3.17):

$$m\left(\frac{\mathrm{d}\boldsymbol{u}_\parallel}{\mathrm{d}t}\right)\cdot\hat{\boldsymbol{B}} \approx q\boldsymbol{E}_\parallel - M\frac{\partial B}{\partial s}\hat{\boldsymbol{B}}. \tag{4.6}$$

Here the longitudinal electric field $\boldsymbol{E}_\parallel$ is determined by equation (2.24). Under the present conditions (4.6) represents a one-dimensional case. The associated Hamiltonian is given in a first approximation by equations (3.60), (3.61), (3.62) and (3.63). For a small electric field and for slow changes in the magnetic field it becomes (cf. GARDNER [1959]):

$$H_\parallel = q\phi + q\alpha\frac{\partial\beta}{\partial t} + \tfrac{1}{2}mu_\parallel^2 + MB, \tag{4.7}$$

where we have introduced the variables (ρ, t) instead of the variables $(\varepsilon\rho, \varepsilon t)$ of equations (3.52) and (3.53) and have returned to the definition of α and β given in Ch. 2, § 1.2. The influence of a gravitation potential ϕ_g can easily be included, but will be left out here.

For the results (4.6) and (4.7) to be valid it is essential that the transverse force field $\boldsymbol{F}_\perp$ is of first and not of zero order in ε. If it would be of zero order, a contribution from the transverse drift $\boldsymbol{E}\times\boldsymbol{B}/B^2$ has to be added to equations (4.6) and (4.7). The longitudinal motion can then no longer be treated in terms of a simple one-dimensional model and cannot be separated from the transverse motion.

In a first approximation equations (4.6), (4.7) and (2.24) show that the longitudinal motion corresponds to a *one-dimensional* system with the canonical coordinates s and $p_\parallel = mu_\parallel = m(\mathrm{d}s/\mathrm{d}t)$, *provided* that M can be considered as a constant. Then, $MB(s)$ behaves as a potential energy. As seen from equations (4.6), (2.24) and (4.7) $H_\parallel - \tfrac{1}{2}mu_\parallel^2$ must have a minimum on

a field line with respect to s if longitudinal oscillations are to occur. We are then allowed to define a separate action integral (2.71) for the longitudinal direction. It becomes

$$J_{\parallel}{}^{*} \equiv J = \oint \left[p_{\parallel} \frac{\partial s}{\partial \vartheta_{\parallel}} \right]_t \mathrm{d}\vartheta_{\parallel} = m \oint u_{\parallel} \, \mathrm{d}s \approx \text{const.}, \tag{4.8}$$

and will be denoted as the longitudinal adiabatic invariant henceforth. Here the longitudinal velocity is defined by $u_{\parallel} = \mathrm{d}s/\mathrm{d}t$ which can have either sign. Equation (4.8) can also be written as

$$J = \oint p_{\parallel} \mathrm{d}s = m \int_0^{t_{\parallel}} u_{\parallel}^2 \mathrm{d}t = m \langle u_{\parallel}^2 \rangle t_{\parallel} = \text{const.}, \tag{4.9}$$

where $\langle u_{\parallel}^2 \rangle$ is the mean value of $u_{\parallel}^2$ during a period $t_{\parallel}$. Observe that s and $p_{\parallel}$ are associated with the generalized coordinates q_2 and p_2 of Ch. 3, § 3.

In the deduction of the guiding centre drift we separated the velocity of gyration from the total velocity of the particle. We now try to proceed in an analogous manner and separate the longitudinal oscillations from the total guiding centre drift. Thus, we should obtain an equation for the mean drift of the guiding centre across the magnetic field lines. For this purpose expand the total time derivative of the equation of motion (3.16) in terms of $\boldsymbol{u}_{\parallel}$ and $\boldsymbol{u}_{\perp}$:

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \left[\frac{\partial}{\partial t} \boldsymbol{u}_{\parallel} + u_{\parallel}^2 (\hat{\boldsymbol{u}}_{\parallel} \cdot \nabla) \hat{\boldsymbol{u}}_{\parallel} + u_{\parallel} \hat{\boldsymbol{u}}_{\parallel} (\hat{\boldsymbol{u}}_{\parallel} \cdot \nabla) u_{\parallel} \right]$$
$$+ \left[\frac{\partial}{\partial t} \boldsymbol{u}_{\perp} + (\boldsymbol{u}_{\perp} \cdot \nabla) \boldsymbol{u}_{\perp} \right] + \left[(\boldsymbol{u}_{\parallel} \cdot \nabla) \boldsymbol{u}_{\perp} + (\boldsymbol{u}_{\perp} \cdot \nabla) \boldsymbol{u}_{\parallel} \right]. \tag{4.10}$$

The first and the last terms within the first square bracket of (4.10) are equal to the left hand member of equation (4.6). We form mean values of (3.16) over a longitudinal period $t_{\parallel}$, with expression (4.10) inserted. A particle will move nearly along a certain field line during a period $t_{\parallel}$. It passes closely to any point on this line and this occurs twice with opposite directions and equal moduli of $\boldsymbol{u}_{\parallel}$ during the same period. Further, $\boldsymbol{u}_{\perp}$ is small and nearly constant at the point just mentioned. Consequently, the mean of the last square bracket of (4.10) vanishes approximately. According to equations (3.16), (4.6) and (4.10) the equation of motion of the mean drift therefore becomes

$$m\left\langle\left[\frac{\partial}{\partial t}+(\boldsymbol{u}_\perp\cdot\nabla)\right]\boldsymbol{u}_\perp\right\rangle + m\langle u_\parallel^2(\hat{\boldsymbol{B}}\cdot\nabla)\hat{\boldsymbol{B}}\rangle$$
$$= q\langle \boldsymbol{E}_\perp + \boldsymbol{u}_\perp \times \boldsymbol{B}\rangle - \langle M\nabla_\perp B\rangle. \qquad (4.11)$$

We now compare equations (2.36), (3.16) and (4.11). The equation of motion (3.16) of the guiding centre is obtained from the exact equation of motion (2.36) by a mean value formation over a Larmor period t_g. This yields an additional mean force, $-M\nabla B$, which is the net result of all contributions from the Larmor motion during a period of gyration. Analogously, a mean value is taken of equation (3.16) over the period $t_\parallel$ in order to obtain (4.11). An additional mean force, $-m\langle u_\parallel^2(\hat{\boldsymbol{B}}\cdot\nabla\hat{\boldsymbol{B}})\rangle$, then results from the mean transverse drift motion of (4.11). It arises from the centrifugal force due to the curvature of the magnetic field lines, as experienced by the particle during a longitudinal period $t_\parallel$. Since it contains $u_\parallel^2$ and is obtained as a mean value over $t_\parallel$ one would expect it to become related to the longitudinal invariant of equation (4.9). The terms containing $u_\parallel^2$ and M in (4.11) have similar forms; the first is proportional to the ratio between $u_\parallel^2$ and the radius of curvature of the magnetic field, and the latter to the ratio between W^2 and the transverse characteristic length of the same field.

The changes of the electric and magnetic fields experienced by a particle during a Larmor period are small and the values measured at the guiding centre can be inserted into equation (3.16). Unfortunately, an analogous mean value formation of (4.11) during the period $t_\parallel$ has to be performed along a field line between two mirror points, i.e. over a length which is comparable to the characteristic lengths of the electric and magnetic fields. This makes (4.11) less suitable for exact deductions of the transverse drift from field line to field line. It only gives a rough indication of the way in which the average transverse drift takes place.

In a more stringent approach we have instead to use canonical equations for the average drift from field line to field line, in terms of the magnetic field coordinates α and β. From the present considerations one would expect M and J to appear as parameters in such a theory. In fact, it will also be shown in § 1.4 that a canonical theory of this kind can be developed, *provided* that the adiabatic invariance of M *and* J is established.

The average transverse drift around configurations of the type sketched in Figures 4.1c and 4.6 may become nearly periodic, as will be shown later in § 1.5. A particle then returns closely to its starting point after a time $t_\perp$ when

it has drifted one turn around the configuration. To lowest order the average transverse drift can then be represented by a *one-dimensional* system of canonical coordinates $\alpha \approx \varepsilon p_1$ and $\beta \approx \varepsilon q_1$ given in Ch. 3, § 3. The corresponding action integral (2.71) is

$$J_\perp^* = \oint \alpha \mathrm{d}\beta = \oint \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} \equiv \Phi \approx \text{const.} \tag{4.12}$$

according to (2.21) and with the gauge $\nabla\chi = 0$. This implies that we should expect the magnetic flux Φ to be constant when the particle revolves around the configuration at the velocity $\boldsymbol{u}_\perp$, provided that the magnetic field changes slowly during a period $t_\perp$ of revolution of $\boldsymbol{u}_\perp$. It is therefore obvious that the assumptions leading to the constancy of Φ are even more restrictive than those concerning J.

For a symmetric mirror configuration such as that shown in Figure 4.1c, we may consider the transverse drift velocity $\boldsymbol{u}_\perp$ of (3.24) in absence of longitudinal motions and external force fields. Since M is constant, the variations of $\boldsymbol{u}_\perp$ are given by $(\partial B/\partial r)/B$. We study the special case where the magnetic field strength can be approximated by $B(r, t) \approx B(r_0, t)\,(r_0/r)^{c_1}$ at the particle orbit, which is given by $r = r_0$ at time $t = 0$. Here c_1 is a constant which differs from zero. Then, $(\partial B/\partial r)/B$ and the drift velocity $u_\perp$ become proportional to $1/r$ and

$$\langle u_\perp^2 \rangle\, t_\perp \approx \text{const.} \tag{4.13}$$

For a slightly asymmetric field the perimeter of the orbit and the velocity $\boldsymbol{u}_\perp$ should only be changed little compared to the case of Figure 4.1c, and (4.13) should at least hold as a rough approximation.

We finally recall attention to the fact that equations (4.5), (4.9) and (4.13) represent similar relations between (W, t_g), $(u_\parallel, t_\parallel)$ and $(u_\perp, t_\perp)$. They are analogous to conditions for the changes of angular momentum and momentum of a mass point which is attached to a wire and revolves around an axis or which bounces between two moving pistons. These results are also intimately connected with the effects of magnetic compression to which we shall return later in Chapter 6.

A summary of the present discussion on adiabatic invariants is presented in Table 4.1.

TABLE 4.1.

Characteristic features of adiabatic invariants for a particle moving in electric and magnetic fields. The latter change at a time scale t_{cf} as seen from a coordinate system following the exact orbit (M) or the mean orbit (J, Φ) of the particle.

Invariants	Associated velocity	Associated period	Approximation holds when	Associated mechanisms
Equivalent magnetic moment M	Velocity of gyration W	Gyro period $t_g = 2\pi/\lvert\omega_g\rvert$, $W^2 t_g \approx$ const.	$t_g \ll t_{cf}$	Changes of the transverse "thermal" energy by compression or expansion of the Larmor radius
Longitudinal invariant J	Longitudinal drift velocity $u_\parallel$	Longitudinal period of oscillation $t_\parallel$, $\langle u_\parallel{}^2 \rangle\, t_\parallel \approx$ const.	$t_g \ll t_\parallel \ll t_{cf}$ $M \approx$ const.	Changes of the longitudinal energy by compression or expansion between two magnetic mirrors
Flux invariant Φ	Transverse drift velocity $u_\perp$	Period of revolution around configuration $t_\perp$, $\langle u_\perp{}^2 \rangle\, t_\perp \approx$ const.	$t_g \ll t_\parallel \ll t_\perp \ll t_{cf}$, $M \approx$ const., $J \approx$ const.	Transverse compression or expansion of the entire particle orbit or plasma body

1.2. THE EQUIVALENT MAGNETIC MOMENT

We have already proved the invariance of the magnetic moment M in (4.2) by means of the action integral (2.71). Starting instead from the equation (3.16) of motion of the guiding centre we can present a first order proof which perhaps illuminates the physical reasons for the constancy of M more clearly.

Scalar multiplication of (3.16) by $\boldsymbol{u}$ yields

$$\tfrac{1}{2}m \frac{\mathrm{d}}{\mathrm{d}t}(u^2) = \boldsymbol{F}\cdot\boldsymbol{u} - M\boldsymbol{u}\cdot\nabla B \tag{4.14}$$

which can be considered as an energy equation for the drift motion. The last term provides couplings between the energies of the motions $\boldsymbol{W}$, $\boldsymbol{u}_\parallel$ and $\boldsymbol{u}_\perp$. Subtract (4.14) from the energy equation (2.38) of the total velocity $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{W}$. The result is

$$\tfrac{1}{2}m \frac{\mathrm{d}}{\mathrm{d}t}(W^2 + 2\boldsymbol{u}_\perp\cdot\boldsymbol{W}) = \boldsymbol{F}\cdot\boldsymbol{W} + M\boldsymbol{u}\cdot\nabla B. \tag{4.15}$$

Further introduce the form (2.37) for the total force $\boldsymbol{F}$, multiply by $\mathrm{d}t$ and integrate over a gyro period $t_g = 2\pi/|\omega_g|$. Since $|\boldsymbol{u}_\perp\cdot\boldsymbol{W}| \ll W^2$ we have with $\boldsymbol{W} = \mathrm{d}\boldsymbol{a}/\mathrm{d}t$

$$\begin{aligned}\tfrac{1}{2}t_g m \frac{\mathrm{d}W^2}{\mathrm{d}t} &\approx -\int_0^{t_g}\left[\nabla(q\phi + m\phi_g) + q\frac{\partial \boldsymbol{A}}{\partial t}\right]\cdot\boldsymbol{W}\mathrm{d}t + t_g M(\boldsymbol{u}\cdot\nabla)B \\ &= -\oint \nabla(q\phi + m\phi_g)\cdot\mathrm{d}\boldsymbol{a} - q\oint\frac{\partial \boldsymbol{A}}{\partial t}\cdot\mathrm{d}\boldsymbol{a} + t_g M(\boldsymbol{u}\cdot\nabla)B.\end{aligned} \tag{4.16}$$

Here we assume, as in the perturbation theory, that $\boldsymbol{F}$, $\boldsymbol{B}$, $\boldsymbol{u}$ and W^2 change very little during a gyro period. At the same time the radius of gyration $\boldsymbol{a}$ describes a nearly closed orbit. Thus, the first integral of the last member of equation (4.16) vanishes approximately and the second integral of the same member can be written as

$$q\oint\frac{\partial \boldsymbol{A}}{\partial t}\cdot\mathrm{d}\boldsymbol{a} = -|q|\iint \hat{\boldsymbol{n}}\cdot\frac{\partial \boldsymbol{B}}{\partial t}\mathrm{d}S \approx -|q|a^2\frac{\partial B}{\partial t}. \tag{4.17}$$

The sign in front of the integrals is due to the fact that ions circulate in the negative and electrons in the positive direction around the field lines, with respect to the positive direction of $\boldsymbol{B}$.

Equation (4.17) represents a betatron acceleration of the Larmor motion which would occur at a fixed point in space, in absence of the drift motion $\boldsymbol{u}$. Remembering that $M = mW^2/2B$, $a = mW/|q|B = W/|\omega_g|$ and $t_g = 2\pi/|\omega_g|$ we obtain from equations (4.16) and (4.17)

$$\tfrac{1}{2}m\,\frac{\mathrm{d}W^2}{\mathrm{d}t} = M\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\right)B. \tag{4.18}$$

The term $M(\boldsymbol{u}\cdot\boldsymbol{\nabla})B$ in this equation yields a betatron acceleration which would be produced when the particle drifts at the velocity $\boldsymbol{u}$ in a stationary, inhomogeneous magnetic field $\boldsymbol{B}$. As a consequence, the entire right hand member of (4.18) gives the total effect due to the changes of the magnetic field as seen from a coordinate system moving with the velocity $\boldsymbol{u}$. Use the definition of M and the fact that the derivatives following the exact particle motion $\boldsymbol{w}$ and the drift $\boldsymbol{u}$ are nearly equal. We then obtain

$$\frac{\mathrm{d}M}{\mathrm{d}t} \approx 0. \tag{4.19}$$

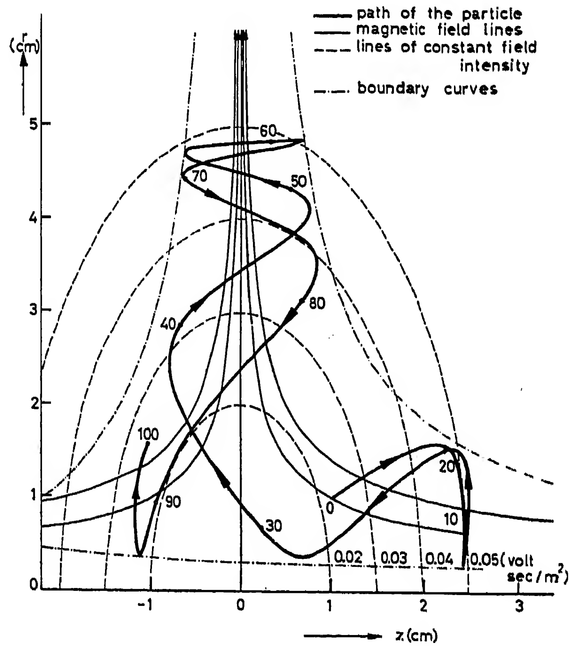


Fig. 4.2. Path of a particle in a magnetic quadrupole field (from BRINKMAN [1960]). Time scale is given by figures at the orbit in 10^{-7} sec.

Consequently, we see from equations (4.18) and (4.19) that the constancy of the magnetic moment M is required by the balance of the energy of gyration. The changes in energy are produced by a two-dimensional adiabatic "compression" of the particle orbit in the magnetic field. The magnetic flux

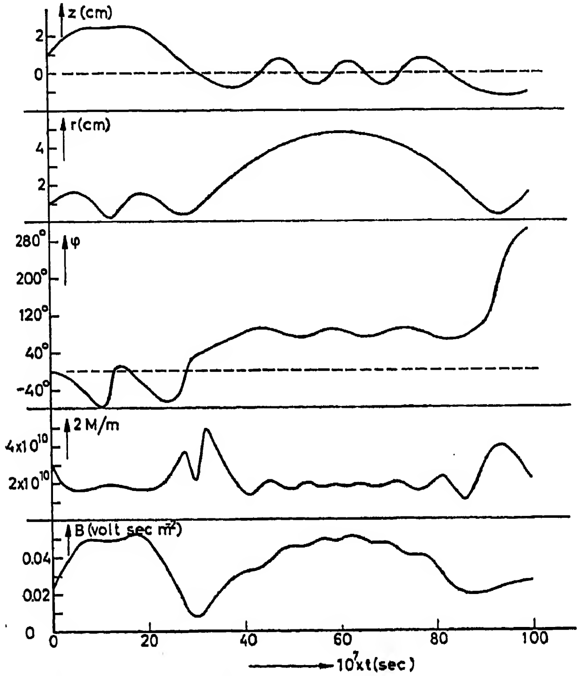


Fig. 4.3. Position, equivalent magnetic moment and local magnetic field strength as functions of time t for the particle of Figure 4.2 (from BRINKMAN [1960]). The Larmor radius is too large for conditions (3.1) and (3.2) to be satisfied.

enclosed by the orbit during a Larmor period is $\pi a^2 B = 2(m/q^2)M$ which is an approximate constant of the motion due to the constancy of M. In other words, the Larmor orbit behaves like a ring-shaped superconductor of flexible and weightless material; it changes its radius in such a way that the enclosed magnetic flux remains constant.

In all physically real cases the Larmor radius is finite as well as the characteristic dimensions and times of the electromagnetic field. The equivalent magnetic moment is then not an exact constant of the motion and the particle orbits are in most cases slightly non-adiabatic. This can be illustrated by the following examples:

(i) Changes of the magnetic moment produced by time variations of a homogeneous magnetic field have been studied by HERTWECK and SCHLÜTER [1957]. Their results will be described further in § 2.

(ii) The motion of a proton in a magnetic quadrupole field, $B_r = c_0 r$; $B_\varphi = 0$; $B_z = -2c_0 z$, has been computed by BRINKMAN [1960]. Figure 4.2 shows the path of the particle at a field strength determined by $c_0 = 1 \text{ V} \cdot \text{sec/m}^3$ and with dimensions r, z of the field given in centimeters. The particle velocity is 3×10^4 m/sec. The variations of the position (r, φ, z) and the magnetic moment M of the particle are shown in Figure 4.3 as functions of time t. The fluctuations of M are large in this example where $a|\nabla B|/B$ is not too far from unity. As a consequence, conditions (3.1) and (3.2) as well as the adiabatic invariance break down.

(iii) The particle containment in an axially symmetric magnetic mirror field has been investigated by GARREN *et al.* [1958] by means of analytic and numerical methods. The residual changes in the magnetic moment between successive mirror reflections are given by these authors as a function of the particle velocity. In particular, the effect of magnetic field imperfections by deviations from symmetry was found to produce particle losses out of the mirror ends. This occurred in a way not consistent with the constancy of M (see also Ch. 7, § 4.1).

*1.3. THE LONGITUDINAL INVARIANT

Before starting a detailed study of the longitudinal invariant consider the simple example demonstrated by Figure 4.4. A particle is trapped in a region

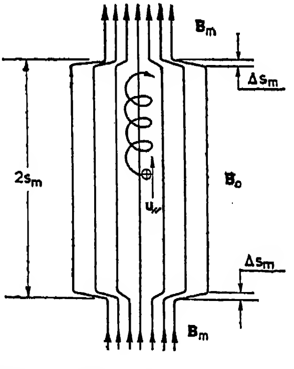


Fig. 4.4. Charged particle trapped between two magnetic mirrors separated at a distance $2\,s_m$ which changes in time.

with the homogeneous magnetic field $\boldsymbol{B}_0$ by two magnetic mirrors at which the field strength changes from $\boldsymbol{B}_0$ to $\boldsymbol{B}_m$. The distance $2s_m$ between the mirrors is altered slowly in time compared to the period of longitudinal oscillations $t_{\parallel}$ of the particle. The latter is reflected in the narrow regions of thickness $\Delta s_m \ll 2s_m$ where the field strength increases from $\boldsymbol{B}_0$ to $\boldsymbol{B}_m$. The situation is analogous to that of an elastic ball which bounces between two moving pistons and gains the velocity $-4\mathrm{d}s_m/\mathrm{d}t$ during each cycle $t_{\parallel} = 4s_m/|u_{\parallel}|$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\,|\,u_{\parallel}\,| = -\left(|u_{\parallel}|/s_m\right)\,\frac{\mathrm{d}s_m}{\mathrm{d}t} \tag{4.20}$$

and

$$4|u_{\parallel}|s_m = u_{\parallel}^2 t_{\parallel} = \text{const.} \tag{4.21}$$

which has the same form as equation (4.9). The result is a one-dimensional adiabatic compression which increases the longitudinal energy, $\frac{1}{2}mu_{\parallel}^2$, as suggested by FERMI [1954].

From the simple situation of Figure 4.4 we now have to proceed to the general case treated by NORTHROP and TELLER [1960]. In the configuration of Figure 4.4 the positions of the reflection points are easily imagined but become less obvious in a magnetic field of a more general shape with smooth

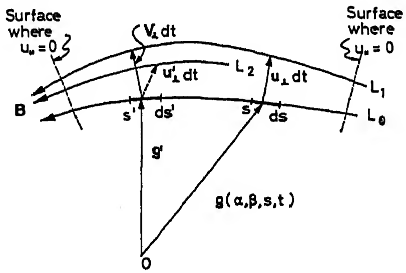


Fig. 4.5. Particle at arc element $\mathrm{d}s$ on a field line L_0 drifting towards the field line L_1 in the time $\mathrm{d}t$ (from NORTHROP and TELLER [1960]).

variations in the longitudinal direction. In addition to this, and what is even more important, a particle in an arbitrary field will drift across the latter at the velocity $\boldsymbol{u}_{\perp}$ during the oscillations. Thus, a particle in the position $\rho(\alpha, \beta, s, t)$ at time t on the field line L_0 of Figure 4.5 will momentarily drift towards the field line L_1. This occurs during the short time interval $\mathrm{d}t = \mathrm{d}s/u_{\parallel}$ which it would spend at the line element $\mathrm{d}s$ on L_0 in absence of a transverse drift $\boldsymbol{u}_{\perp}$. If the particle would instead be situated at $\boldsymbol{\rho}'$ it would drift from $\mathrm{d}s'$

towards a field line L_2 which is *not* necessarily the same as L_1. Because of this slow and variable drift off a line of force during a longitudinal period of oscillation there is no strict analogy between the general case and the one-dimensional motion of Figure 4.4. Thus, we shall now develop an analysis which considers the fact that the particle does not return exactly to the same point after one longitudinal period $t_\parallel$.

It will be shown later that we can uniquely define an action integral (4.8) for each given field line. Due to the transverse drift from one field line to another there will be a momentarily change $\mathrm{d}J/\mathrm{d}t$ of the longitudinal invariant J as seen by an observer moving along the actual path of the particle. It is therefore obvious that $\mathrm{d}J/\mathrm{d}t$ will differ from zero; we are not going to prove that the instantaneous value of J is constant. What will be shown is instead that the mean value

$$\left\langle \frac{\mathrm{d}J}{\mathrm{d}t} \right\rangle = \frac{1}{t_\parallel} \oint \frac{\mathrm{d}J}{\mathrm{d}t} \cdot \frac{\mathrm{d}s}{u_\parallel} \tag{4.22}$$

taken over a whole longitudinal period $t_\parallel$ vanishes approximately. This implies that the sum of all changes of J occuring at different positions ρ and ρ' should nearly cancel during a period $t_\parallel$.

Assume $\boldsymbol{F}$ to arise from an electric field $\boldsymbol{E}$ and introduce the Hamiltonian $H_\parallel$ of the longitudinal motion given by equation (4.7). As we have seen from § 1.1 a definition of a longitudinal invariant J makes only sense when the equivalent magnetic moment $M = mW^2/2B$ is an adiabatic invariant. Assume this to be the case in the discussion which follows. We also restrict ourselves to the case where oscillations take place in the space between two magnetic mirrors such that $H_\parallel - \frac{1}{2}mu_\parallel^2$ has a minimum between the latter. The terms appearing in (4.7) vary slowly compared to the gyro time t_g. The total time derivative $\dot{H}_\parallel$ is therefore nearly equal to the derivative following the drift velocity $\boldsymbol{u}$:

$$\begin{aligned} \dot{H}_\parallel &\approx \left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla\right) H_\parallel \\ &\approx \frac{\partial}{\partial t}\left(q\phi + q\alpha \frac{\partial \beta}{\partial t} + MB\right) + q\boldsymbol{u}_\perp \cdot \left(\frac{\partial \beta}{\partial t} \nabla\alpha - \frac{\partial \alpha}{\partial t} \nabla\beta\right), \end{aligned} \tag{4.23}$$

where $\nabla(\phi + \alpha \cdot \partial\beta/\partial t)$ has been substituted from (2.23), equation (4.14) has been used to eliminate $\boldsymbol{E} \cdot \boldsymbol{u}$ and we remember that $\nabla\alpha$ and $\nabla\beta$ are perpendicular to $\boldsymbol{B}$. A small term $(u_\perp/u_\parallel)^2$ has also been neglected compared to

unity in the present deduction. The form (4.23) contains no partial derivative with respect to the longitudinal direction s. Therefore $\partial H_{\parallel}/\partial s = 0$, which can also be obtained from the longitudinal component (4.6) of the guiding centre equation of motion in combination with equation (4.7). If we move along a field line (α = const., β = const.) with the longitudinal velocity $u_{\parallel}$ of the particle we measure a rate of change in $H_{\parallel}$ given by the first term of the right hand member of equation (4.23). For a stationary field $\dot{H}_{\parallel}$ would vanish and $H_{\parallel}$ would correspond to the energy integral of the longitudinal oscillations in a potential "trough" given by $q\phi + MB$. When $\partial H_{\parallel}/\partial t$ differs from zero $H_{\parallel}$ still represents the energy integral of the same motion. $H_{\parallel}$ is also still constant along a field line when the oscillations are rapid compared to the time scale of the external field, but $H_{\parallel}$ then changes slowly in time with the latter.

With the present definitions the longitudinal invariant becomes

$$J = m \oint u_{\parallel}(s')\mathrm{d}s', \quad u_{\parallel} = (2/m)^{\frac{1}{2}}\left(H_{\parallel} - q\phi - q\alpha\frac{\partial\beta}{\partial t} - MB\right)^{\frac{1}{2}}, \tag{4.24}$$

where integration should take place along all elements $\mathrm{d}s'$ of a given field line L_0; the functions $H_{\parallel}$, ϕ, $\alpha(\partial\beta/\partial t)$, and MB are uniquely determined at each time t for every line L_0. When the particle moves, both changes in $u_{\parallel}$ and $\mathrm{d}s'$ in equation (4.24) will contribute to $\mathrm{d}J/\mathrm{d}t$. These changes must be calculated not only for $\mathrm{d}s$ but for all other arcs $\mathrm{d}s'$ on L_0 between the reflection points to give the total contribution to $\mathrm{d}J/\mathrm{d}t$. At any arc $\mathrm{d}s'$ on L_0 we define the velocity $\boldsymbol{V}_{\perp}(s, s')$ which is perpendicular to $\hat{\boldsymbol{B}}(s')$ and carries a point from L_0 to L_1 in the same time $\mathrm{d}t$ that the actual particle on $\mathrm{d}s$ goes from L_0 to L_1. Observe that $\boldsymbol{V}_{\perp}$ differs from the drift velocity $\boldsymbol{u}'_{\perp}$ at $\mathrm{d}s'$ as seen in Figure 4.5; when the particle actually arrives at $\mathrm{d}s'$, it will not be drifting towards L_1, but towards another line L_2.

We are now in a position to give a detailed definition of the instantaneous value of $\mathrm{d}J/\mathrm{d}t$. When the particle starts at $\mathrm{d}s$ on L_0 it will drift to L_1 after the time $\mathrm{d}t$. The integral (4.24) then changes from its uniquely determined value $J(\mathrm{L}_0)$ on L_0 to the corresponding value $J(\mathrm{L}_1)$ obtained by integration along L_1. With $u_{\parallel}' \equiv u_{\parallel}(s')$ the change can be written as

$$\begin{aligned} \mathrm{d}J &= m \oint \left[\mathrm{d}u'_{\parallel}\mathrm{d}s' + u'_{\parallel}\mathrm{d}(\mathrm{d}s')\right] \\ &= m \oint (1/2u'_{\parallel})\left[2u'^{2}_{\parallel}\mathrm{d}(\mathrm{d}s') + \mathrm{d}(u'^{2}_{\parallel})\mathrm{d}s'\right]. \end{aligned} \tag{4.25}$$

Thus, the instantaneous change $\mathrm{d}J$ during the time $\mathrm{d}t$ at the point p in Figure 4.5 is due to changes in $(u'_\parallel)^2$ as well as in the arc length $\mathrm{d}s'$ during the drift from one field line to another.

First consider the changes in $(u'_\parallel)^2$. For the contribution from $H_\parallel$ we observe that $H_\parallel$ is constant on a field line. When the particle moves from L_0 to L_1 during the time $\mathrm{d}t$ this contribution to the entire integral (4.25) arises from $\mathrm{d}H_\parallel = \dot{H}_\parallel(s)\mathrm{d}t$ with $\dot{H}_\parallel$ given by equation (4.23). For the rest of the terms of $u'^2_\parallel$ defined by the last of expressions (4.24) we have

$$\mathrm{d}\left[q\phi + q\alpha\frac{\partial\beta}{\partial t} + MB\right]' = \left(\frac{\partial}{\partial t} + \boldsymbol{V}_\perp\cdot\nabla\right)\left(q\phi + q\alpha\frac{\partial\beta}{\partial t} + MB\right)'\mathrm{d}t, \quad (4.26)$$

where a prime indicates that they are evaluated at s'. These contributions vary along L_0. They are given at each point s' by the time derivative including the velocity $\boldsymbol{V}_\perp(s, s')$ which carries the points on L_0 to L_1, according to the definitions. Finally, we obtain a change in the arc length during the time $\mathrm{d}t$ from the rate of change of the radius R of curvature as seen in the frame of reference which follows the particle:

$$\mathrm{d}(\mathrm{d}s') = \boldsymbol{V}_\perp\cdot\nabla(\mathrm{d}s')\mathrm{d}t = -\boldsymbol{V}_\perp\cdot(\partial\hat{\boldsymbol{B}}/\partial s')\mathrm{d}s'\mathrm{d}t. \quad (4.27)$$

Equation (4.27) is easily derived from geometric considerations (cf. Fig. 3.3).

Summing up the results (4.26), (4.27) and (4.25) the rate of change of J becomes

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \oint\left[\dot{H}_\parallel - \left(\frac{\partial}{\partial t} + \boldsymbol{V}_\perp\cdot\nabla\right)\left(q\phi + q\alpha\frac{\partial\beta}{\partial t} + MB\right)' - mu'^2_\parallel\boldsymbol{V}_\perp\cdot\left(\frac{\partial\hat{\boldsymbol{B}}}{\partial s}\right)'\right]\frac{\mathrm{d}s'}{u'_\parallel}. \quad (4.28)$$

The vector $\boldsymbol{V}_\perp$ must now be evaluated explicitly. It represents a line-preserving velocity field determined by the local drift $\boldsymbol{u}_\perp(s)$ at p in Figure 4.5. Since (α, β) determine the position of a field line we have

$$\dot{\alpha}(s) = (\partial\alpha/\partial t + \boldsymbol{u}_\perp\cdot\nabla\alpha)_s = (\partial\alpha/\partial t + \boldsymbol{V}_\perp\cdot\nabla\alpha)_{s'}, \quad (4.29)$$

$$\dot{\beta}(s) = (\partial\beta/\partial t + \boldsymbol{u}_\perp\cdot\nabla\beta)_s = (\partial\beta/\partial t + \boldsymbol{V}_\perp\cdot\nabla\beta)_{s'}, \quad (4.30)$$

where $\boldsymbol{V}_\perp(s' = s) = \boldsymbol{u}_\perp(s)$. We have also defined $\boldsymbol{V}_\perp$ to be perpendicular to the field lines:

$$\hat{\boldsymbol{B}}(s')\cdot\boldsymbol{V}_\perp(s') = 0 \quad (4.31)$$

at all points s' on L_0. The solution for $\boldsymbol{V}_\perp$ is

$$\boldsymbol{V}_\perp = \left[\left(\dot{\alpha}\nabla\beta' - \dot{\beta}\nabla\alpha'\right) + \left(\frac{\partial\beta}{\partial t}\nabla\alpha - \frac{\partial\alpha}{\partial t}\nabla\beta\right)'\right]\times(\boldsymbol{B}/B^2)', \quad (4.32)$$

where a prime denotes evaluation at s'. The result (4.32) is immediately seen to satisfy condition (4.31). From straight-forward vector calculations we obtain

$$\boldsymbol{V}_\perp \cdot \nabla\alpha' = \left(\dot{\alpha} - \frac{\partial \alpha'}{\partial t}\right)(\nabla\beta \times \boldsymbol{B}/B^2)' \cdot \nabla\alpha' = \dot{\alpha} - \partial\alpha'/\partial t, \tag{4.33}$$

the last member being deduced by the help of equation (2.22). This result satisfies (4.29) and from the symmetry of α and β we also see that (4.30) becomes satisfied.

We now use these equations to rewrite the integral (4.28). According to (3.18), (3.19) and (3.20) the transverse drift is

$$\boldsymbol{u}_\perp = \left(q\boldsymbol{E} - M\nabla B - mu_\parallel^2 \frac{\partial \hat{\boldsymbol{B}}}{\partial s} - m\frac{\mathrm{d}\boldsymbol{u}_\perp}{\mathrm{d}t}\right) \times \boldsymbol{B}/qB^2, \tag{4.34}$$

where the last term within the bracket can be neglected in a first approximation. With the expressions (2.23) for $\boldsymbol{E}$, (2.22) for $\boldsymbol{B}$, (4.32) for $\boldsymbol{V}_\perp$ and (4.34) for $\boldsymbol{u}_\perp$ the integral (4.28) becomes after some straightforward deductions

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \oint\Bigg[\dot{H}_\parallel(s) - \frac{\partial}{\partial t}\left(q\phi + q\alpha\frac{\partial\beta}{\partial t} + MB\right)' - q\boldsymbol{u}_\perp' \cdot \left(\frac{\partial\beta}{\partial t}\nabla\alpha - \frac{\partial\alpha}{\partial t}\nabla\beta\right)'$$
$$- q\dot{\alpha}\left(\frac{\partial\beta}{\partial t} + \boldsymbol{u}_\perp\nabla\beta\right)' + q\dot{\beta}\left(\frac{\partial\alpha}{\partial t} + \boldsymbol{u}_\perp \cdot \nabla\alpha\right)'\Bigg]\frac{\mathrm{d}s'}{u_\parallel'}. \tag{4.35}$$

Here $\dot{\alpha}$ and $\dot{\beta}$ are evaluated at s. Comparison with eq. (4.23) now shows that the second and third terms of the integrand in (4.35) are equal to $\dot{H}_\parallel$ evaluated at s':

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \oint\left\{\dot{H}_\parallel(s) - \dot{H}_\parallel(s') + q\left[\dot{\beta}(s)\dot{\alpha}(s') - \dot{\alpha}(s)\dot{\beta}(s')\right]\right\}\frac{\mathrm{d}s'}{u_\parallel'} \tag{4.36}$$

with the notations $\dot{\alpha}$, $\dot{\beta}$ used in equations (4.29) and (4.30).

The result (4.36) shows that there is no reason for $\mathrm{d}J/\mathrm{d}t$ to vanish in general. However, with the definition (4.22) we have

$$\left\langle\frac{\mathrm{d}J}{\mathrm{d}t}\right\rangle = (1/t_\parallel)\oint\oint\Big\{\dot{H}_\parallel(s) - \dot{H}_\parallel(s')$$
$$+ q\left[\dot{\beta}(s)\dot{\alpha}(s') - \dot{\alpha}(s)\dot{\beta}(s')\right]\Big\}\frac{\mathrm{d}s\mathrm{d}s'}{u_\parallel u_\parallel'} = 0. \tag{4.37}$$

Thus, it has been proved that the average rate of change of J vanishes because of the antisymmetry of the integrand of equation (4.37).

The present conclusions agree with those drawn directly from the action integral (4.8) in the discussion of § 1.1. The advantage with the proof given by (4.37) is that it takes into consideration the instantaneous drift $\boldsymbol{u}_\perp$ off the field lines during the longitudinal oscillations. This makes the restrictions less severe than those imposed on J in § 1.1 where we had to start with the assumption that the particle remains on a field line during the period $t_\parallel$. The present proof also gives insight into the mechanism by which J is conserved. The antisymmetry of the integrand of (4.37) shows that the contributions to the change of J by transverse drifts at different points of a field line cancel over a period $t_\parallel$.

*1.4. EQUATIONS OF MOTION FOR THE AVERAGE DRIFT

When the oscillations along a magnetic field line are fast compared to the transverse drift, one will be primarily interested in the average motion which transfers the particle from line to line, i.e., in the changes of α and β. This problem has earlier been investigated by KADOMTSEV [1958] and by NORTHROP and TELLER [1960].

The variables specifying the state of the average drift are α, β, the energy integral $H_\parallel$ of the longitudinal motion, and the magnetic moment M which represents the energy of gyration. Therefore we can consider $J = J(\alpha, \beta, H_\parallel, M, t)$ as a function of these quantities which are constant on a field line; α, β, and $H_\parallel$ will change during the average drift in the transverse direction. Equation (4.36) can be written as

$$\frac{\mathrm{d}J}{\mathrm{d}t} = t_\parallel(\dot{H}_\parallel - \langle \dot{H}_\parallel \rangle) - qt_\parallel(\dot{\alpha}\langle\dot{\beta}\rangle - \dot{\beta}\langle\dot{\alpha}\rangle), \tag{4.38}$$

where $\langle\ \rangle$ denotes average values taken over a period $t_\parallel$ of oscillation. We can also write

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \frac{\partial J}{\partial t} + \frac{\partial J}{\partial \alpha}\dot{\alpha} + \frac{\partial J}{\partial \beta}\dot{\beta} + \frac{\partial J}{\partial H_\parallel}\dot{H}_\parallel \tag{4.39}$$

for the total changes in $(\alpha, \beta, H_\parallel)$ space since M is constant for a given particle. Comparison between equations (4.38) and (4.39) yields

$$\langle \dot{H}_\parallel \rangle = -\frac{1}{t_\parallel}\frac{\partial J}{\partial t}, \tag{4.40}$$

$$\langle\dot{\beta}\rangle = -\frac{1}{qt_{\|}}\frac{\partial J}{\partial \alpha}, \tag{4.41}$$

$$\langle\dot{\alpha}\rangle = \frac{1}{qt_{\|}}\frac{\partial J}{\partial \beta}, \tag{4.42}$$

$$1 = \frac{1}{t_{\|}}\frac{\partial J}{\partial H_{\|}}. \tag{4.43}$$

The last of these equations is obvious from the definition (4.24) of J. The first three are the required equations of motion of the average transverse drift with the longitudinal motion eliminated. On the average the particle moves towards that adjacent line on which J is unchanged.

Equations (4.40) – (4.43) are not of canonical form because the factor $t_{\|}$ is a function of $(\alpha, \beta, H_{\|}, M, t)$. However, we can solve the relation $J = J(\alpha, \beta, H_{\|}, M, t)$ for $H_{\|}$ and write $H_{\|} = H_{\|}(\alpha, \beta, J, M, t)$. By implicit differentiation of J and $H_{\|}$ and by substituting the expression for dJ into d$H_{\|}$ we obtain $\partial J/\partial\alpha = -(\partial H_{\|}/\partial\alpha)/(\partial H_{\|}/\partial J)$, etc. Then equations (4.40) – (4.43) become

$$\langle\dot{\alpha}\rangle = -\frac{1}{q}\frac{\partial H_{\|}}{\partial \beta}, \tag{4.44}$$

$$\langle\dot{\beta}\rangle = \frac{1}{q}\frac{\partial H_{\|}}{\partial \alpha}, \tag{4.45}$$

$$\langle\dot{H}_{\|}\rangle = (\partial H_{\|}/\partial t), \tag{4.46}$$

$$1 = t_{\|}(\partial H_{\|}/\partial J). \tag{4.47}$$

Equations (4.44), (4.45) and (4.46) are of canonical form and correspond directly to equations (2.57), (2.56) and (2.59).

The equations for the rate of change of α and β can be written in vector form. For an observer at a particular point s' in Figure 4.5 the particle is "seen" to drift towards the line L_1 when it is on ds at s. This can be conceived as a fictive drift $\boldsymbol{V}_{\perp}(s, s')$ from L_0 to L_1 at s'. The mean velocity by which the particle is seen at s' to move away from L_0 to another field line is made up by the contributions from all actual positions s of the particle during the period $t_{\|}$. Therefore this velocity becomes equal to

$$\langle\boldsymbol{V}_{\perp}\rangle = \frac{1}{t_{\|}}\oint \boldsymbol{V}_{\perp}(s, s')\frac{\mathrm{d}s}{u_{\|}} \tag{4.48}$$
$$= \left[\langle\dot{\alpha}\rangle\nabla\beta' - \langle\dot{\beta}\rangle\nabla\alpha' + \left(\frac{\partial\beta}{\partial t}\nabla\alpha - \frac{\partial\alpha}{\partial t}\nabla\beta\right)'\right] \times (\boldsymbol{B}/B^2)'$$

according to equation (4.32). Substitution of $\langle\dot{\alpha}\rangle$ and $\langle\dot{\beta}\rangle$ from equations (4.44) and (4.45) yields

$$\langle \boldsymbol{V}_{\perp}\rangle = \boldsymbol{B} \times \left[\nabla H_{\|} - q\left(\frac{\partial\beta}{\partial t}\nabla\alpha - \frac{\partial\alpha}{\partial t}\nabla\beta\right)\right]\Big/ qB^2, \tag{4.49}$$

where we have dropped the primes and remember that $H_{\|}$ is constant on a field line.

With the methods of Ch. 2, § 1.3 we can now investigate whether the field $\langle \boldsymbol{V}_{\perp}\rangle$ is flux-preserving. Consider

$$\begin{aligned} &\operatorname{curl}(\boldsymbol{E} + \langle \boldsymbol{V}_{\perp}\rangle \times \boldsymbol{B}) \qquad\qquad (4.50)\\ &= -\frac{\partial}{\partial t}(\nabla\alpha \times \nabla\beta) + \frac{1}{q}\operatorname{curl}\left[\nabla H_{\|} - q\left(\frac{\partial\beta}{\partial t}\nabla\alpha - \frac{\partial\alpha}{\partial t}\nabla\beta\right)\right]\\ &= -\frac{\partial}{\partial t}(\nabla\alpha \times \nabla\beta) - \nabla\left(\frac{\partial\beta}{\partial t}\right) \times \nabla\alpha + \nabla\left(\frac{\partial\alpha}{\partial t}\right) \times \nabla\beta = 0, \end{aligned}$$

where use has been made of equations (2.1), (2.22) and (4.49). The condition (2.29) for flux-preservation is therefore fulfilled by the velocity field $\langle \boldsymbol{V}_{\perp}\rangle$. According to (2.34) the same field is then also line-preserving which is consistent with its definition in § 1.3 of this chapter.

Let $Q_{\mathrm{p}}(\alpha, \beta, J, M, t)$ be the particle density in (α, β, J, M) space at time t. Each point in this space represents a particle somewhere on the line (α, β) at time t with magnetic moment M and longitudinal invariant J. Motions in the space describe the average motion of the particles when mean values have been formed over both the gyration and the longitudinal oscillations. Since $\langle\dot{J}\rangle$ and $\dot{M}$ vanish the equation of continuity in this space becomes

$$\frac{\partial Q_{\mathrm{p}}}{\partial t} + \frac{\partial}{\partial\alpha}(Q_{\mathrm{p}}\langle\dot{\alpha}\rangle) + \frac{\partial}{\partial\beta}(Q_{\mathrm{p}}\langle\dot{\beta}\rangle) = 0. \tag{4.51}$$

By equations (4.44) and (4.45) $(\partial/\partial\alpha)\ \langle\dot{\alpha}\rangle = -\ (\partial/\partial\beta)\ \langle\dot{\beta}\rangle$ so that

$$\frac{\mathrm{d}Q_{\mathrm{p}}}{\mathrm{d}t} = \frac{\partial Q_{\mathrm{p}}}{\partial t} + \langle\dot{\alpha}\rangle\frac{\partial Q_{\mathrm{p}}}{\partial\alpha} + \langle\dot{\beta}\rangle\frac{\partial Q_{\mathrm{p}}}{\partial\beta} = 0 \tag{4.52}$$

which implies that the density Q_{p} remains constant during the mean motion at the velocities $\langle\dot{\alpha}\rangle$ and $\langle\dot{\beta}\rangle$. This is a Liouville theorem in (α, β, J, M) space. A theorem of this kind exists for every canonical system (see also Ch. 5, § 1.1).

When we use orthogonal coordinates in the $\alpha\beta$ plane a surface element

will represent the magnetic flux $\mathrm{d}\Phi = \mathrm{d}\alpha\mathrm{d}\beta$ and $Q_\mathrm{p}\mathrm{d}\alpha\mathrm{d}\beta$ is the corresponding number of particles of moment M and longitudinal invariant J.

Especially, in a stationary state substitution of expressions (4.44) and (4.45) into equation (4.52) yields

$$\frac{\partial H_{\|}}{\partial \alpha} \cdot \frac{\partial Q_\mathrm{p}}{\partial \beta} - \frac{\partial H_{\|}}{\partial \beta} \cdot \frac{\partial Q_\mathrm{p}}{\partial \alpha} = 0. \tag{4.53}$$

Equations (4.44), (4.45) and (4.46) are of canonical form with (α, β) as generalized coordinates and momenta and $H_{\|}$ as the corresponding Hamiltonian. Thus, we see from (2.59) that Q_p becomes a constant of the motion on a *longitudinal invariant surface*, i.e., on a surface of fixed J, M and $H_{\|}$. Additional discussions on such surfaces will be made in § 1.5 of this chapter.

Let us use the obtained results to consider the particle density $n(\rho, H_{\|}, M, t)$ in configuration space. If a steady state exists along a given field line we assume the density to be given by

$$n = n_1 B/u_{\|} \tag{4.54}$$

which varies inversely as $u_{\|}$ and is inversely proportional to the cross-sectional area of a flux tube; n_1 is a quantity to be determined and which is constant on the field line. It is obtained from integration between the reflection points:

$$\int \frac{n\mathrm{d}s}{B} = \tfrac{1}{2} n_1 t_{\|}. \tag{4.55}$$

A flux tube of cross sectional area $\mathrm{d}S = \mathrm{d}\alpha\mathrm{d}\beta/B$ and length $\mathrm{d}s$ contains $n\mathrm{d}S\mathrm{d}s$ particles. The total number of particles in this tube and between the reflection points is defined as $N_\mathrm{f}(\alpha, \beta, H_{\|}, M, t)\mathrm{d}\alpha\mathrm{d}\beta$. Then, $N_\mathrm{f} = \frac{1}{2}n_1 t_{\|}$ according to equation (4.55). The number of particles with magnetic moment M in the element $\mathrm{d}\alpha\mathrm{d}\beta$ and with values of the longitudinal invariant between J and $J + \mathrm{d}J$ can either be written as $Q_\mathrm{p}\mathrm{d}J\mathrm{d}\alpha\mathrm{d}\beta$ or as $N_\mathrm{f}\mathrm{d}H_{\|}\mathrm{d}\alpha\mathrm{d}\beta$, where $\mathrm{d}J$ and $\mathrm{d}H_{\|}$ are associated values. The quantities N_f and Q_p are therefore related by $N_\mathrm{f}\mathrm{d}H_{\|} = Q_\mathrm{p}\mathrm{d}J$, or $N_\mathrm{f} = Q_\mathrm{p}\partial J/\partial H_{\|}$. Further, according to equation (4.43), $N_\mathrm{f}/t_{\|} = Q_\mathrm{p}$. From this (4.54) becomes

$$n = 2(B/u_{\|})Q_\mathrm{p}. \tag{4.56}$$

Because Q_p is constant on a longitudinal invariant surface in a steady state this result shows that n becomes proportional to $B/u_{\|}$ on such a surface. In absence of electric fields $u_{\|}$ depends only on B at given values of J, M and $H_{\|}$. Thus, contours of constant B on an invariant surface are then also contours of constant density n.

* 1.5. THE FLUX INVARIANT

An average over the gyro period of the exact equation of motion results in the guiding centre equation (3.16), which contains the adiabatic invariant M. In its turn, the average of the guiding centre equation over the longitudinal oscillations results in the equations of motion (4.44) and (4.45) of the average drift with the invariant J as parameter. Finally, it will now be shown that an average of the equations of the average drift will lead to a third invariant Φ, provided that the motion in $(\alpha, \beta, H_{\|})$ space is periodic.

First consider a static magnetic field configuration of slightly distorted symmetry, somewhat like that surrounding the earth (Fig. 4.6). In absence

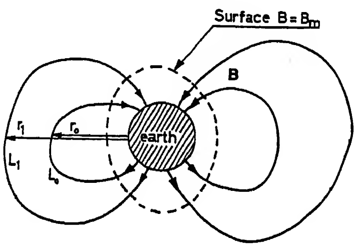


Fig. 4.6. Earth's magnetic field imagined as a dipole-like, slightly asymmetric field with surface of a certain field strength B_m indicated in figure (cf. NORTHROP and TELLER [1960]).

of additional force fields the particle energy and total velocity w_0 remain constant according to equation (2.38). Constancy of the magnetic moment $M = mW^2/2B$ then leads to

$$u_{\|}^2 = w_0^2(1 - B/B_m), \tag{4.57}$$

where B_m is the magnetic field strength at the reflection points of the longitudinal oscillations. As the particle drifts around the configuration of Figure 4.6 the reflection points will appear on the surface $B = B_m$ of the figure. For a dipole field the mean value of $u_{\|}$ along a field line obtained from (4.57) increases with the equatorial distance r_0 of the line. Since the arc length between the reflection points at $B = B_m$ increases as well, we see that J increases with r_0 if we move from a field line L_0 to L_1 where $r_1 > r_0$. If there is only a slight asymmetry in the magnetic field a qualitatively similar situation is encountered. With NORTHROP and TELLER [1960] we conclude that the lines L_0 and L_1 correspond to different values of the longitudinal invariant J. Due to the constancy of J the particle must therefore return to

the same field line in Figure 4.6 when it has drifted one turn around the configuration. This takes place in a time $t_\perp$.

Thus, we have proved that, at least in the present stationary case, the mean transverse motion around the configuration is nearly periodic. The closed surface which the particle traces out in this way for a given value of J is a longitudinal invariant surface.

Further, when the magnetic field changes slowly compared to the time $t_\perp$ of revolution the system will be nearly periodic with respect to the average drift described by equations (4.44) – (4.47). In (α, β, s) space this drift traces out a cylindrical surface perpendicular to the $\alpha\beta$ plane as shown in Figure 4.7. The area enclosed in the $\alpha\beta$ plane is the magnetic flux Φ enclosed by the particle orbit in configuration space. The height of the cylindrical surface at a certain position α, β in (α, β, s) space is equal to the arc length along a field line between two reflection points. As the particle moves in configuration space the corresponding point in (α, β, s) space oscillates along the vertical ends of the cylinder of Figure 4.7 and drifts around its surface at

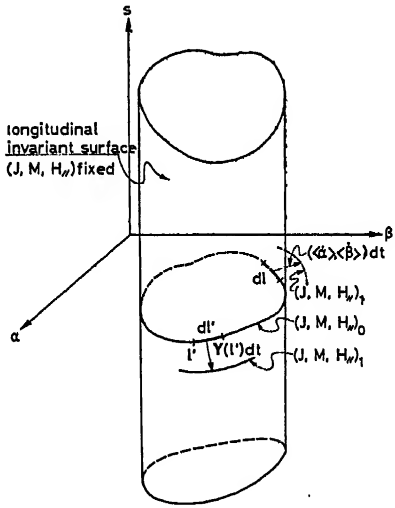


Fig. 4.7. Motion on a longitudinal invariant surface in (α, β, s) space.

the same time. We shall now show that the enclosed flux Φ remains constant when $t_\perp$ is small compared to the time scale of the changes in the magnetic field. This implies that the slow drifts of the particle in (α, β, s) space at

right angles to the longitudinal invariant surface will cancel during a period $t_\perp$, somewhat like the cancellation leading to $\langle \mathrm{d}J/\mathrm{d}t \rangle = 0$ as discussed earlier in this chapter.

Suppose that at some time the particle is on $\mathrm{d}l$ in Figure 4.7 drifting slowly at right angles to $\mathrm{d}l$ while it moves rapidly around the contour in the $\alpha\beta$ plane. We denote the invariant surface where J, M and $H_\parallel$ have their values at the "starting point" l by $(J, M, H_\parallel)_0$. We also define a velocity $\boldsymbol{Y}(l, l')$ perpendicular to $\mathrm{d}l'$ which carries a point at l' on the longitudinal invariant surface $(J, M, H_\parallel)_0$ to a new surface $(J, M, H_\parallel)_1$ in the same time $\mathrm{d}t$ that the actual particle at l moves from $(J, M, H_\parallel)_0$ to $(J, M, H_\parallel)_1$ with the velocity $(\langle \dot{\alpha} \rangle, \langle \dot{\beta} \rangle)$. The velocity $\boldsymbol{Y}$ is analogous to $\boldsymbol{V}_\perp$ in Figure 4.5. The values of J and M are the same on the surfaces indicated by $(_0)$ and $(_1)$ in Figure 4.7, but not those of $H_\parallel$.

Since J and M are approximately constant and $H_\parallel$ is independent of s the time derivative of $H_\parallel$ at l during the average motion becomes

$$\langle \dot{H}_\parallel \rangle_l = \frac{\partial}{\partial t} H_\parallel(l') + \boldsymbol{Y}(l, l') \cdot \boldsymbol{\nabla}_{\alpha\beta} H_\parallel(l') \tag{4.58}$$

according to the definition of $\boldsymbol{Y}$. Here $\boldsymbol{\nabla}_{\alpha\beta}$ denotes the gradient in the $\alpha\beta$ plane. But $H_\parallel$ is constant on the closed curve of Figure 4.7 and $\boldsymbol{\nabla}_{\alpha\beta} H_\parallel(l')$ is therefore perpendicular to the line element $\mathrm{d}l'$. The instantaneous rate of change of the enclosed area (and of the magnetic flux) then becomes equal to the sum of all contributions from displacements $\boldsymbol{Y}(l, l')\mathrm{d}t$ at different positions l' which are associated with the instantaneous motion at l:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \oint \frac{\boldsymbol{Y}(l, l') \cdot \boldsymbol{\nabla}_{\alpha\beta} H_\parallel(l')}{|\boldsymbol{\nabla}_{\alpha\beta} H_\parallel(l')|} \,\mathrm{d}l'. \tag{4.59}$$

Substituting $\boldsymbol{Y} \cdot \boldsymbol{\nabla}_{\alpha\beta} H_\parallel$ from equation (4.58), $V_{\alpha\beta} \equiv [\langle \dot{\alpha} \rangle^2 + \langle \dot{\beta} \rangle^2]^{\frac{1}{2}}$ from equations (4.44) and (4.45) and $\partial H_\parallel / \partial t$ from (4.46) into (4.59) we obtain

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \oint [\langle \dot{H}_\parallel \rangle_l - \langle \dot{H}_\parallel \rangle_{l'}] \frac{\mathrm{d}l'}{V'_{\alpha\beta}}, \tag{4.60}$$

where the primes denote evaluation at l'. Thus, the instantaneous change in the flux Φ does not vanish. However, for a complete cycle around the longitudinal invariant surface the mean of $\mathrm{d}\Phi/\mathrm{d}t$ becomes

$$\left\langle \frac{\mathrm{d}\Phi}{\mathrm{d}t} \right\rangle = \oint\oint [\langle \dot{H}_\parallel \rangle_l - \langle \dot{H}_\parallel \rangle_{l'}] \frac{\mathrm{d}l \mathrm{d}l'}{V_{\alpha\beta} V'_{\alpha\beta}} = 0 \tag{4.61}$$

due to the antisymmetry of the integrand. This implies that the contributions of the drifts off the invariant surface cancel during a period $t_\perp$. The cancellation is analogous to that found earlier for the longitudinal invariant J.

Equation (4.61) completes the proof that the flux Φ is an adiabatic invariant during the average motion. It is also connected with the fact that ionized matter is nearly "frozen" to the lines of force of a slowly changing magnetic field.

2. Higher Order Invariance: One-Dimensional Problem

In the preceding paragraph we have seen that certain adiabatic invariants can be defined which are constant at least in first order. In a real physical situation the Larmor radius and the Larmor period are not infinitely small compared to the changes of the electromagnetic field in space and time and the defined quantities are no exact constants. To investigate how large the deviations from constancy may become, it is therefore necessary to consider higher order approximations. In response to this HELLWIG [1955] proved the constancy of the equivalent magnetic moment to second order. KULSRUD [1957] considered the problem of the harmonic oscillator with slowly changing elasticity and found that its adiabatic invariant was constant to all orders. The analogous result was proved by KRUSKAL [1958] for a gyrating particle and by LENARD [1959] for the anharmonic oscillator. GARDNER [1959] has established the adiabatic invariance for any simply-periodic classical system to all orders, and has applied the results also to the longitudinal invariant J.

In this paragraph we restrict ourselves to the one-dimensional case; an extension to three dimensions will be undertaken in § 3 of this chapter. The present discussions on the one-dimensional problem are connected with the papers by HERTWECK and SCHLÜTER [1957], CHANDRASEKHAR [1958], and BROER and VAN WIJNGAARDEN [1959].

2.1. DEFINITIONS OF ADIABATIC INVARIANCE

The one-dimensional problem which we are faced with corresponds to the equation

$$\ddot{\zeta} + \omega^2(t)\zeta = 0 \tag{4.62}$$

for an oscillating system. The coordinate $\zeta(t)$ represents the deviation from the equilibrium and $\omega(t)$ is the instantaneous frequency of oscillation. We have already mentioned one example of this kind in Ch. 2, § 4.4 where a

particle gyrates in a homogeneous, time-dependent magnetic field and $\omega^2 = \frac{1}{4}\omega_g^2$. Another example is given by a harmonic oscillator with slowly changing elasticity, such as a pendulum the suspending thread of which is being changed gradually. Finally, the longitudinal oscillations of a particle along the axis of a slowly varying magnetic mirror field lead to an equation of the form (4.6) with $E_{\|} = 0$. It corresponds to (4.62), provided that the magnetic field $B(s)$ is proportional to s^2 and the restoring force, $-M\partial B/\partial s$, becomes proportional to s for $M =$ const. Without this restriction on $B(s)$ the situation becomes analogous to that of the anharmonic oscillator.

For systems which can be described by (4.62) we will now define the adiabatic invariance in a way suggested by CHANDRASEKHAR [1958]. Study a transition from a state where $\omega(t) \to \omega_0$ as $t \to -\infty$ to a state where $\omega(t) \to \omega_f$ as $t \to +\infty$. We also assume the derivatives $d^\nu\omega/dt^\nu$ to vanish for all $\nu \geqslant 1$ in the limits $t \to -\infty$ and $t \to +\infty$. In these states ω is constant and the solutions of (4.62) are simply

$$\zeta(t \to -\infty) = C_{10} \exp(-i\omega_0 t) + a_{10} \exp(i\omega_0 t),$$

$$\zeta(t \to +\infty) = C_{1f} \exp(-i\omega_f t) + a_{1f} \exp(i\omega_f t), \qquad (4.63)$$

where C_{10}, C_{1f}, a_{10} and a_{1f} are complex constants.

For a one-dimensional harmonic oscillator the displacements are given by ζ and the kinetic and potential energies are represented by $|\dot{\zeta}|^2$ and $\omega^2|\zeta|^2$ the mean values of which are equal during a period of oscillation. Each of these quantities is also equal to the half of the total energy. The latter is therefore proportional to $\omega^2 \langle|\zeta|^2\rangle$. The adiabatic invariant which we are going to discuss is in this case equal to the ratio between the energy and the frequency and is proportional to $\omega\langle|\zeta|^2\rangle$.

For longitudinal oscillations of a particle between two magnetic mirrors in a field $B(s) \propto = \zeta^2 s^2$ the same ratio becomes proportional to $\langle u_{\|}{}^2\rangle t_{\|}$ where $u_{\|} = d\zeta/dt$ is the longitudinal velocity and $t_{\|}$ is the period of oscillation. It is therefore proportional to the longitudinal invariant J of equation (4.9) and to $\omega\langle|\zeta|^2\rangle$ where $\omega = 2\pi/t_{\|}$.

Finally, for a particle gyrating in a homogeneous, time-dependent magnetic field the definitions in Ch. 2, § 4.4 and equations (4.63) imply that the transverse motion of the particle is given by (4.63) and by

$$\zeta_1 = x + iy = \zeta \exp[i(\omega t + \text{const.})] \quad (t \to \pm\infty) \qquad (4.64)$$

in the initial and final states. Then C_{10}, C_{1f} and a_{10}, a_{1f} will correspond

to the position of the guiding centre and to the Larmor radius, respectively. From (4.64) and Chapter 2 follows that the equivalent magnetic moment is equal to the ratio between the kinetic energy of the Larmor motion and the frequency, i.e.,

$$M_0 = (q/2\omega_0)\,(\dot{x}^2 + \dot{y}^2)_0 = (q/2\omega_0)\,\langle|\dot{\zeta}_1|^2\rangle_0 \tag{4.65}$$

in the initial state. A similar relation holds for the final state. Here we observe that $|\dot{\zeta}_1|^2 = |\dot{\zeta} + i\omega\zeta|^2$ becomes equal to $4\omega_0^2 a_{10}^2$ and $4\omega_f^2 a_{1f}^2$ in the initial and final states, respectively. Therefore $(1/\omega)\langle|\dot{\zeta}_1^2|\rangle$ corresponds to ωa^2, where the Larmor radius a can be considered as a "mean displacement" of the particle with respect to the centre of gyration. In particular, if we can neglect the motions of the centre of gyration and put $C_{10} = C_{1f} = 0$ in (4.63), we see that $(1/\omega)\langle|\dot{\zeta}_1|^2\rangle$ becomes equal to $\omega\langle|\zeta|^2\rangle$ in the case of a gyrating particle.

In the examples mentioned here the expressions

$$\Lambda = \frac{\omega_f\langle|\zeta|^2\rangle_{t\to+\infty}}{\omega_0\langle|\zeta|^2\rangle_{t\to-\infty}} \quad\text{and}\quad \Lambda = \frac{(1/\omega_f)\langle|\dot{\zeta}_1|^2\rangle_{t\to+\infty}}{(1/\omega_0)\langle|\dot{\zeta}_1|^2\rangle_{t\to-\infty}} \tag{4.66}$$

become a measure of the deviation from adiabatic invariance. The former of equations (4.66) is suited for considerations of the simple harmonic oscillator, whereas the latter has a more convenient form for the discussion of a charged particle in a magnetic field. We shall treat specific problems of the latter in §§ 2.2 and 2.3.

The deviation Λ clearly depends on the manner in which $\omega(t)$ varies between ω_0 and ω_f. Let $d\omega/dt$ be bounded and define the characteristic time

$$\left|\frac{\omega}{d\omega/dt}\right|_{\min} = t_\omega. \tag{4.67}$$

A precise statement of the theorem on adiabatic invariance would then be the assertion that Λ approaches 1 when $t_\omega \to \infty$, i.e., when the changes of the external parameters become infinitely slow. This assertion is not restricted by any limitation on ω_f/ω_0. Nevertheless it will be convenient to distinguish two cases:

(i) When the relative change in frequency of the transition is small we can define

$$\delta = \tfrac{1}{2}|(\omega_f - \omega_0)/(\omega_f + \omega_0)| \ll 1. \tag{4.68}$$

If it then can be shown that $A = 1 + O(\delta^2)$ as $t_\omega \to \infty$ we shall say that we have adiabatic invariance "in the small".

(ii) If A approaches 1 when $t_\omega \to \infty$ with no restriction on ω_f/ω_0 we say that we have adiabatic invariance "in the large". This can also be formulated somewhat differently. Let $\omega \equiv \omega(\varepsilon t)$ represent a one-parameter family of time variations satisfying the requirements specified at the beginning of this paragraph. Then, A will depend on ε and adiabatic invariance in the large can be defined by stating that $A(\omega_0, \omega_f, \varepsilon t)$ should approach 1 as ε tends to zero, independently of ω_0 and ω_f.

The proof of the adiabatic invariance in the small for a system described by (4.62) can easily be accomplished. Let

$$t_1 = \int \omega \mathrm{d}t, \quad \mathrm{d}t_1 = \omega \mathrm{d}t \tag{4.69}$$

and (4.62) becomes

$$\frac{\mathrm{d}^2\zeta}{\mathrm{d}t_1^2} + \zeta = -\left(\frac{1}{\omega}\frac{\mathrm{d}\omega}{\mathrm{d}t_1}\right)\frac{\mathrm{d}\zeta}{\mathrm{d}t_1}. \tag{4.70}$$

This equation can be solved by an iteration procedure, where the right hand member is evaluated in terms of the solution obtained when this member is ignored. Of the solutions of (4.70) for $t_1 \to -\infty$, where the right hand member vanishes, we chose $\zeta = \text{const.} \cdot \exp.\,(it_1)$. Put this solution into the right hand member of equation (4.70). From the equation which then follows we take the solution which tends to $\exp(it_1)$ when $t_1 \to -\infty$ and $\omega \to \omega_0$:

$$\zeta = \left(1 - \tfrac{1}{2}\log\frac{\omega}{\omega_0}\right)\exp(it_1) + P(t_1)\exp(-it_1). \tag{4.71}$$

Here we have introduced

$$P(t_1) = \tfrac{1}{2}\int_{-\infty}^{t_1} \frac{1}{\omega}\frac{\mathrm{d}\omega}{\mathrm{d}t_1'}\exp(2it_1')\,\mathrm{d}t_1'. \tag{4.72}$$

From the average of $|\zeta|^2$ over all initial phases we obtain for the final state as $t_1 \to +\infty$:

$$\lim_{t_1 \to +\infty}\left\{\omega\langle|\zeta|^2\rangle\right\} = \omega_f\left(1 - \tfrac{1}{2}\log\frac{\omega_f}{\omega_0}\right)^2 + \omega_f|P(\infty)|^2, \tag{4.73}$$

with the notation $P(\infty) \equiv P(t_1 = +\infty)$. From the first of relations (4.66) we now obtain

$$A = \frac{\omega_f}{\omega_0}\left(1 - \tfrac{1}{2}\log\frac{\omega_f}{\omega_0}\right)^2 + \frac{\omega_f}{\omega_0}|P(\infty)|^2. \tag{4.74}$$

When the total change in frequency, $\delta\omega = \omega_f - \omega_0$, is small as stated in condition (4.68) we have $O[(\delta\omega)^2] = O(\delta^2)$ and

$$\Lambda = 1 + O(\delta^2) + \frac{\omega_f}{\omega_0} |P(\infty)|^2. \tag{4.75}$$

In the limit of infinitely slow variations, where t_ω of (4.67) approaches zero, the contribution from $P(\infty)$ in (4.75) vanishes and Λ has the form required for adiabatic invariance in the small.

Adiabatic invariance in the large is more difficult to establish. Introduce the variable $\zeta_2 = \sqrt{\omega}\zeta$ and (4.70) becomes

$$\frac{d^2\zeta_2}{dt_1^2} + \zeta_2 = -G_1\zeta_2, \tag{4.76}$$

where

$$G_1(t_1) = \frac{1}{4}\left(\frac{1}{\omega}\frac{d\omega}{dt_1}\right)^2 - \frac{1}{2\omega}\frac{d^2\omega}{dt_1^2}. \tag{4.77}$$

Observe that G_1 includes the changes of ω during the transition and that these are of second order in $d\omega/dt_1$, in contrast to equation (4.70). We try to solve (4.76) by an iteration process in the same manner as we have treated equation (4.70). The obtained solution which tends to $\exp(it_1)$ in the initial state where $t_1 \to -\infty$ is

$$\zeta_2 = \exp(it_1) + \tfrac{1}{2}i\left\{\exp(it_1)\int_{-\infty}^{t_1} G_1(t_1')dt_1' - \exp(-it_1)\int_{-\infty}^{t_1} G_1(t_1')\exp(2it_1')dt_1'\right\}. \tag{4.78}$$

For slow variations of the parameters we now write $\varepsilon t_1 = \tau_1$ instead of t_1 and investigate the limit $\varepsilon \to 0$. When $t_1 \to +\infty$ in the final state the solution (4.78) approaches the value

$$\zeta_2(+\infty) = \exp(it_1) + \tfrac{1}{2}i\varepsilon\exp(it_1)\int_{-\infty}^{+\infty} G_1(\tau_1)d\tau_1 - \tfrac{1}{2}i\varepsilon\exp(-it_1)\int_{-\infty}^{+\infty} G_1(\tau_1)\exp(2i\tau_1/\varepsilon)d\tau_1, \tag{4.79}$$

where $G_1(\tau_1)$ is the function (4.77) with t_1 replaced by τ_1. If the integrals contained in (4.79) exist, then it follows immediately from the same equation that $\zeta_2 \to \exp(it_1)$ also in the final state, $t_1 \to +\infty$, provided that $\varepsilon \to 0$.

Thus, $A(\omega_0, \omega_f, \varepsilon t)$ approaches 1 as ε tends to zero and the adiabatic invariance in the large follows.

The present treatment suffices to establish the adiabatic invariance in the large, but does not yield the error term with exactitude for $\varepsilon \to 0$. The reason for this is that the iteration process which gives the solution (4.78) is not uniformly convergent for the entire range of t_1 from $-\infty$ to $t+\infty$. We shall return to the estimation of the error term in the specific example of § 2.3.

*2.2. DISCONTINUOUS MAGNETIC FIELD

The deviations from adiabatic invariance can be deduced explicitly in some special cases. As a first example we choose the problem treated by HERTWECK and SCHLÜTER [1957], where a charged particle moves in a homogeneous field which jumps discontinuously at time $t = 0$ from $B_0 = m\omega_0/q$ to $B_f = m\omega_f/q$. Rewrite (2.138) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\zeta_1}{\mathrm{d}t} + i\omega\zeta_1\right) - \tfrac{1}{2}i\frac{\mathrm{d}\omega}{\mathrm{d}t}\zeta_1 = 0 \tag{4.80}$$

which is integrated to

$$\frac{\mathrm{d}\zeta_1}{\mathrm{d}t} + i\omega\zeta_1 - \tfrac{1}{2}i\int^t \frac{\mathrm{d}\omega}{\mathrm{d}t}\zeta_1 \mathrm{d}t = c_1, \tag{4.81}$$

where c_1 is a constant. Since ζ_1 represents the position of the particle which cannot change during the infinitely short time when ω jumps from ω_0 to ω_f, we immediately obtain from equation (4.81)

$$\frac{\mathrm{d}\zeta_1}{\mathrm{d}t} + i\omega_0\zeta_1 = c_1 \quad (t < 0) \tag{4.82}$$

and

$$\frac{\mathrm{d}\zeta_1}{\mathrm{d}t} + i\omega_f\zeta_1 - \tfrac{1}{2}i(\omega_f - \omega_0)\zeta_1(0) = c_1 \quad (t > 0). \tag{4.83}$$

Here $\zeta_1(0)$ is the value of ζ_1 at $t = 0$.

Now calculate A defined in the last of equations (4.66). Since ω is constant except at $t = 0$ also $|\dot{\zeta}_1|^2$ is constant except at $t = 0$, where ζ_1 is continuous and has the value $\zeta_1(0)$. The deviation of A from 1 produced by the jump in ω is then easily calculated from the values of $\mathrm{d}\zeta_1/\mathrm{d}t$ of equations (4.82) and (4.83) at $t = -0$ and $t = +0$. Especially for $c_1 = 0$ the initial state is given by $\zeta_1(t \to -\infty) = \exp(-i\omega_0 t)$ and corresponds to a particular phase with respect to the jump. We then obtain

$$\Lambda = \frac{(\omega_0 + \omega_f)^2}{4\omega_0\, \omega_f} = 1 + \frac{(\omega_0 - \omega_f)^2}{4\omega_0\, \omega_f}. \tag{4.84}$$

Due to the jump in ω at $t = 0$ condition (3.2) is violated and the adiabatic invariance is destroyed.

*2.3. SLOWLY VARYING MAGNETIC FIELD

As we have seen from the discussion on adiabatic invariance in the large in § 2.1 of the present chapter, the magnetic moment M is strictly a constant in a homogeneous magnetic field B only in the limit of infinitely slow field variations. To investigate how large the deviations from constancy are for a slow but finite variation of B we use a method developed by HERTWECK and SCHLÜTER [1957]. Introduce a new variable y_1 defined by

$$\zeta = \exp\left[i \int \frac{y_1 + 1}{y_1 - 1}\, \mathrm{d}t_1\right]. \tag{4.85}$$

Equation (4.70) then becomes

$$\frac{\mathrm{d}y_1}{\mathrm{d}t_1} - 2iy_1 + \frac{1}{2\omega}\frac{\mathrm{d}\omega}{\mathrm{d}t_1}(1 - y_1^2) = 0, \tag{4.86}$$

or

$$\dot{y}_1 - 2i\omega y_1 + \frac{1}{2}\frac{\dot{\omega}}{\omega}(1 - y_1^2) = 0, \tag{4.87}$$

if we introduce $\mathrm{d}t_1 = \omega \mathrm{d}t$ according to (4.69) and a dot indicates time derivative. The magnetic field is assumed to stay constant during the initial period $-\infty < t \leqslant t_0$ and during the final period $t_f \leqslant t < +\infty$. It varies during the interval $t_0 < t < t_f$.

We now restrict the discussion to small values of y_1^2 compared to unity. This is reasonable since the solution of (4.70) in a homogeneous field is $\zeta = \text{const.} \cdot \exp(\pm it_1)$ and expression (4.85) should deviate only little from this form in a slowly varying field. Neglecting the term y_1^2 the general solution of (4.87) becomes

$$y_1(t) = -P(t_0, t) \exp\left(2i \int_{t_0}^{t} \omega \mathrm{d}t'\right), \quad y_1(t_0) = 0, \tag{4.88}$$

$$P(t_0, t) = \tfrac{1}{2} \int_{t_0}^{t} \left(\frac{\dot{\omega}}{\omega}\right)_{t'} \exp\left(-2i \int_{t_0}^{t'} \omega \mathrm{d}t''\right) \mathrm{d}t'. \tag{4.89}$$

From equations (4.88), (4.89), (4.85) and (2.139) we can calculate $\dot{\zeta}_1$ in

terms of the obtained solution for $y_1^2 \ll 1$. We shall not present the details of the calculation but refer to the original work by HERTWECK and SCHLÜTER [1957]. A mean value formation over all possible phases of the initial state determined by $\zeta_1(t \to -\infty)$ can be shown to give

$$\Lambda = 1 + [1 + c_1^2]|P(-\infty, +\infty)|^2 \tag{4.90}$$

for the transition from the initial to the final state. In (4.90) c_1 is a constant of order unity defined in the original paper.

The integral $P(-\infty, +\infty)$ is of the type given by the Fourier transformation. We substitute t_1 by $\tau_1 = \varepsilon t_1$ and study slow field variations (ε small) as suggested in § 2.1. Assume $\omega(t)$ to start from a constant level ω_0 and to increase to a new level ω_f as indicated in Figure 4.8. From (4.89) we have

$$P(\varepsilon) \equiv P(-\infty, +\infty) = \tfrac{1}{2} \int_{-\infty}^{+\infty} f(u_1) \exp(-iu_1/\varepsilon)\, du_1 \tag{4.91}$$

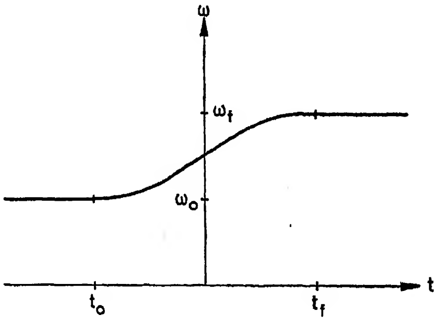


Fig. 4.8. The variation of ω with time (HERTWECK and SCHLÜTER [1957])

with

$$f(u_1) = \left[\frac{d\omega}{d(\varepsilon t)} \Big/ \omega^2(\varepsilon t)\right]_{t(u_1)} \tag{4.92}$$

and

$$u_1 = \int_0^{\varepsilon t} \omega(\varepsilon t')d(\varepsilon t'), \quad du_1/d(\varepsilon t') = \omega > 0. \tag{4.93}$$

With the present form (4.91) of $P(\varepsilon)$ it is difficult to judge what happens in the limit $\varepsilon \to 0$. However, suppose that $f(u_1 + iv_1)$ is an analytic and integrable function in the complex u_1v_1 plane within a strip $|v_1| < v_{1m}$. We can then make the transformation $u_1 = u_1' - i\varepsilon_m$, where $0 < \varepsilon_m < v_{1m}$ and ε_m is a constant. This implies that the path of integration of equation

(4.91) is displaced from the axis $v_1 = 0$ to the line $v_1 = \varepsilon_m$. Since f is integrable within the region we obtain

$$|P(\varepsilon)| = \tfrac{1}{2}\exp(-\varepsilon_m/\varepsilon)\left|\int_{-\infty}^{+\infty} f(u_1' - i\varepsilon_m)\exp(-iu_1'/\varepsilon)du_1'\right| \leqslant$$

$$\leqslant \mathscr{I}_{\max}\exp(-\varepsilon_m/\varepsilon), \quad (4.94)$$

where $2\mathscr{I}_{\max}$ is the upper limit of the integral (4.91). If we choose instead $\varepsilon_m < 0$ a relation similar to (4.94) is obtained the right hand member of which diverges in the limit $\varepsilon \to 0$. From such a relation no conclusion can be

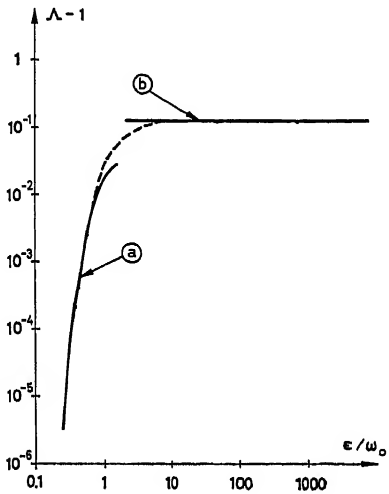


Fig. 4.9. The deviation $\Lambda - 1$ from adiabatic invariance as a function of the relative rapidity ϵ/ω_0 of the magnetic field variation. Curve (a) is calculated for slow variations according to § 2.3 and curve (b) for rapid variations according to § 2.2. The dashed part of the curve is due to numerical solution of the exact equation of motion (HERTWECK and SCHLÜTER [1957]).

drawn about the magnitude of $|P(\varepsilon)|$, but it does not contradict the result (4.94) for $\varepsilon_m > 0$. The latter shows that $|P(\varepsilon)|$ tends to zero at least as fast as $\exp(-\varepsilon_m/\varepsilon)$.

The quantity $\varepsilon_m/\varepsilon$ is essentially the ratio between the gyro time and the characteristic time of the magnetic field. Therefore the result (4.94) and equation (4.90) show that the deviations from adiabatic invariance become very small as soon as this ratio is much less than unity. A detailed calculation

has been made by HERTWECK and SCHLÜTER [1957] for the special case where $c_1 = 0$ in equation (4.90) and

$$\omega(t) = \tfrac{1}{2}\omega_0[3 + \tanh(\varepsilon t)]. \tag{4.95}$$

The result is shown in Figure 4.9 which gives the deviation $\Lambda - 1$ from adiabatic invariance as a function of the relative rapidity ε/ω_0 of the magnetic field variation. The branch (a) is obtained from the calculations of slow variations in the present paragraph, whereas the branch (b) is determined from the treatment of a discontinuous variation in § 2.2. The deviation $\Lambda - 1$ becomes very small as soon as $\varepsilon < \omega_0$.

A theory which gives better approximations to the present problem has been developed by CHANDRASEKHAR [1958].

It should finally be mentioned that TAMOR [1960] has extended the investigations to include singularities in the restoring force of the one-dimensional oscillator.

3. Higher Order Invariance: Three-Dimensional Problem

During the motion in an electromagnetic field of general shape the particle will drift along and across the magnetic field lines. It is not immediately clear that adiabatic invariance can then be studied by the methods of § 2 which apply to the one-dimensional case. As we have seen earlier, there is a coupling between the gyration and the longitudinal and transverse drift motions. However, it has been shown by GARDNER [1959] and demonstrated in Ch. 3, § 3 that a canonical transformation can be applied to the exact equations of motion. The result of this is that the particle motion is described by three degrees of freedom which essentially represent the gyration and the longitudinal and transverse drift motions. In this new representation GARDNER [1959] found that it is possible to study the adiabatic invariance by methods similar to those applied to the one-dimensional harmonic oscillator. A detailed description of these results is beyond the scope of the present volume. Instead we shall restrict ourselves in this paragraph to a treatment of the equivalent magnetic moment by the action integral as suggested by KRUSKAL [1957].

The development (3.6) of the position vector $\rho(t)$ of the particle has been shown by BERKOWITZ and GARDNER [1959] to satisfy the exact equation of motion (3.3). The action integral (2.71) for a nearly periodic system can then be defined and has the form (4.2). This is so, because the solution (3.6) admits the introduction of a function $\vartheta(t)$ associated with the gyration. Therefore the action integral J^* of (4.2) is an exact constant. As a conse-

quence, we will now see that the integrand of (4.2) can be developed in a way to give an expansion of the equivalent magnetic moment to all orders in the "smallness parameter" ε. The result is obtained by substitution of $\boldsymbol{\rho}$ and $\mathrm{d}\boldsymbol{\rho}/\mathrm{d}t$ from (3.6) into the integral (4.2).

Now return to the results of Ch. 3, § 1.3. The sum of the scalar product between $\boldsymbol{C}_1$ and (3.32) and between $\boldsymbol{S}_1$ and (3.33) gives the relation

$$
\begin{aligned}
&(C_1^2 + S_1^2)\dot{\vartheta}^2 + 2\dot{\vartheta}(\boldsymbol{C}_1 \times \boldsymbol{S}_1)\cdot\boldsymbol{B} + 2\varepsilon\dot{\vartheta}(\boldsymbol{S}_1\cdot\dot{\boldsymbol{C}}_1 - \boldsymbol{C}_1\cdot\dot{\boldsymbol{S}}_1)\\
&\quad + \varepsilon\boldsymbol{C}_1\cdot(\dot{\boldsymbol{C}}_1 \times \boldsymbol{B}) + \varepsilon\boldsymbol{S}_1\cdot(\dot{\boldsymbol{S}}_1 \times \boldsymbol{B}) - \varepsilon B(C_1^2 + S_1^2)\Delta\omega = \mathrm{O}(\varepsilon^2),
\end{aligned}
\tag{4.96}
$$

where the expression (3.34) for the force $\boldsymbol{F}$ has been substituted into equations (3.32) and (3.33) and we have defined

$$
\begin{aligned}
B(C_1^2 + S_1^2)\Delta\omega = {}& \dot{\boldsymbol{C}}\cdot[\boldsymbol{C}_1 \times (\boldsymbol{C}_1\cdot\nabla)\boldsymbol{B} + \boldsymbol{S}_1 \times (\boldsymbol{S}_1\cdot\nabla)\boldsymbol{B}]\\
&+ \boldsymbol{C}_1\cdot[(\boldsymbol{C}_1\cdot\nabla)(\dot{\boldsymbol{C}} \times \boldsymbol{B})] + \boldsymbol{S}_1\cdot[(\ {}_1\cdot\nabla)(\dot{\boldsymbol{C}} \times \boldsymbol{B})].
\end{aligned}
\tag{4.97}
$$

The coefficients are evaluated at the centre of gyration.

Relations (3.35) and (3.36) are satisfied by

$$
\boldsymbol{C}_1 = -\,\boldsymbol{S}_1 \times \hat{\boldsymbol{B}} + \mathrm{O}(\varepsilon), \quad \boldsymbol{S}_1 = \boldsymbol{C}_1 \times \hat{\boldsymbol{B}} + \mathrm{O}(\varepsilon) \tag{4.98}
$$

in conformity with equation (3.39). As was pointed out in Ch. 3, § 1.3 in connexion with (3.37) the higher order terms of $\dot{\vartheta}$ are not fixed by the expansion (3.6) and can be chosen freely. The form of these terms depends upon the way in which we prefer to represent the vectors $\boldsymbol{C}_\nu$ and $\boldsymbol{S}_\nu$ connected with the gyration (BRINKMAN [1959]). Impose the condition

$$
\boldsymbol{C}_1\cdot\dot{\boldsymbol{S}}_1 = \mathrm{O}(\varepsilon). \tag{4.99}
$$

From the time derivative of the last expression (3.35) and from equations (4.99) and (4.98) we then obtain

$$
\boldsymbol{C}_1\cdot\dot{\boldsymbol{S}}_1 = -\,\dot{\boldsymbol{C}}_1\cdot\boldsymbol{S}_1 + \mathrm{O}(\varepsilon) = \boldsymbol{C}_1\cdot(\dot{\boldsymbol{C}}_1 \times \hat{\boldsymbol{B}}) + \mathrm{O}(\varepsilon), \tag{4.100}
$$

and a similar relation can be deduced for $\boldsymbol{S}_1\cdot(\dot{\boldsymbol{S}}_1 \times \hat{\boldsymbol{B}})$. This result implies that we have chosen a representation where $\boldsymbol{C}_1$ and $\dot{\boldsymbol{C}}_1$ are almost parallel as well as $\boldsymbol{S}_1$ and $\dot{\boldsymbol{S}}_1$, i.e., the vectors $\boldsymbol{C}_1$ and $\boldsymbol{S}_1$ do not rotate around $\boldsymbol{B}$ but only their amplitudes vary in time in the first approximation. This is the same choice as HELLWIG [1955] has made. Then the third, fourth and fifth terms of (4.96) vanish according to (4.100) and the solution for $\dot{\vartheta}$ is

$$
\dot{\vartheta} = B + \varepsilon\Delta\omega + \mathrm{O}(\varepsilon^2), \tag{4.101}
$$

which is easily verified by insertion into (4.96) and is a direct consequence of equation (4.99).

We are now in a position to evaluate the action integral of equation (4.2). From the expansion (3.6) we obtain

$$J^* = m\oint\left[\dot{C} + \varepsilon\dot{C}_1\cos\frac{\vartheta}{\varepsilon} + \varepsilon\dot{S}_1\sin\frac{\vartheta}{\varepsilon} - \dot{\vartheta}C_1\sin\frac{\vartheta}{\varepsilon} + \dot{\vartheta}S_1\cos\frac{\vartheta}{\varepsilon} + \frac{q}{m}A + O(\varepsilon^2)\right]_t \tag{4.102}$$

$$\cdot\left[-C_1\sin\frac{\vartheta}{\varepsilon} + S_1\cos\frac{\vartheta}{\varepsilon} - 2\varepsilon C_2\sin\frac{2\vartheta}{\varepsilon} + 2\varepsilon S_2\cos\frac{2\vartheta}{\varepsilon} + O(\varepsilon^2)\right]_t \mathrm{d}\vartheta = \text{const.},$$

where subscript $(_t)$ denotes that t should be kept constant in all places where it occurs explicitly. This implies that derivatives of $\boldsymbol{C}$, $\boldsymbol{C}_1$ and $\boldsymbol{S}_1$ with respect to ϑ vanish when J^* is evaluated. The integration is performed over a period $2\pi\varepsilon$ of ϑ and (4.102) becomes

$$J^* = \varepsilon m\pi\dot{\vartheta}(C_1^2 + S_1^2) \tag{4.103}$$

$$+ q\int_0^{2\pi\varepsilon}\left[A\cdot\left(S_1\cos\frac{\vartheta}{\varepsilon} - C_1\sin\frac{\vartheta}{\varepsilon}\right)\right]_t \mathrm{d}\vartheta + O(\varepsilon^3) = \text{const.},$$

when it is observed that $\dot{\boldsymbol{C}}_1 \cdot \boldsymbol{S}_1$ and $\boldsymbol{C}_1 \cdot \dot{\boldsymbol{S}}_1$ vanish in first order according to equation (4.100). Since the explicit dependence on t is suppressed in the integrand of (4.103) the corresponding integration takes place around a closed orbit. By Stokes' theorem the integral is then given by the magnetic flux encircled by the vector $\boldsymbol{\rho}$ during one period of ϑ. The surface element is given by

$$\hat{n}\mathrm{d}S = -\tfrac{1}{2}\rho \times (\rho + \mathrm{d}\rho) = -\tfrac{1}{2}a \times \frac{\partial a}{\partial \vartheta}\mathrm{d}\vartheta$$

$$= -\tfrac{1}{2}\varepsilon(C_1 \times S_1)\mathrm{d}\vartheta + O(\varepsilon^2), \tag{4.104}$$

where the minus sign indicates that a positively charged particle gyrates in the negative direction with respect to $\boldsymbol{B}$. With the magnetic field given by the expansion (3.7) and with the surface element inserted from (4.104) the integral can be evaluated and (4.103) reduces to

$$J^* = \varepsilon m\pi\dot{\vartheta}(C_1^2 + S_1^2) - \varepsilon m\pi B\cdot(C_1 \times S_1) + O(\varepsilon^3) = \text{const.} \tag{4.105}$$

From equations (4.98) and (3.35) we obtain

$$\hat{B}\cdot(C_1 \times S_1) = \tfrac{1}{2}(C_1^2 + S_1^2) + O(\varepsilon^2) \tag{4.106}$$

and the result (4.105) can be written

$$J^*/2\pi\varepsilon = \tfrac{1}{2}m(\dot{\vartheta} - \tfrac{1}{2}B)(C_1^2 + S_1^2) + O(\varepsilon^2) = \text{const.}, \qquad (4.107)$$

where $J^*/2\pi\varepsilon$ represents a mean value of the integrand of equation (4.102).

The mean value of the square of the gyration velocity $\boldsymbol{W} = \mathrm{d}\boldsymbol{a}/\mathrm{d}t$ over a period $\vartheta = 2\pi\varepsilon$ is easily deduced from the expansion (3.6):

$$\langle W^2\rangle = (1/2\pi\varepsilon)\int_0^{2\pi\varepsilon} (\dot{\boldsymbol{a}})_t\, \mathrm{d}\vartheta = \tfrac{1}{2}\dot{\vartheta}^2(C_1^2 + S_1^2) + O(\varepsilon^2). \qquad (4.108)$$

Combination of equations (4.108) and (4.101) then yields an expression for the equivalent magnetic moment (cf. BRINKMAN [1959]):

$$M = m\langle W^2\rangle/2B = \tfrac{1}{2}m(\dot{\vartheta} - \tfrac{1}{2}B)(S_1^2 + C_1^2) + O(\varepsilon^2). \qquad (4.109)$$

Consequently, equation (4.107) shows that

$$M = \text{const.} + O(\varepsilon^2). \qquad (4.110)$$

We have therefore deduced an explicit expression for M including first order contributions from ε. The result shows that deviations from the adiabatic invariance of M for finite values of ε are limited at least to terms of second order.

In principle, the present evaluations of J^* can be continued by the help of expressions (3.6) and (4.2) to any order of ε. This has been done by KRUSKAL [1958] who has deduced a power series in ε for the instantaneous value of $[\hat{\boldsymbol{B}} \times (\boldsymbol{w} \times \hat{\boldsymbol{B}}) - \boldsymbol{u}_E]^2/B$, with $\boldsymbol{w} = \mathrm{d}\rho/\mathrm{d}t$ given by equation (3.6). As a result, this quantity becomes an adiabatic invariant to all orders. Further considerations about the invariance concept will be made in the next paragraph.

An extension of the definition of the first adiabatic invariant to the relativistic and three-dimensional case has recently been presented by VANDERVOORT [1960]. He has applied the methods of § 2.1 earlier introduced by CHANDRASEKHAR [1958].

4. General Discussion on Invariance Concept

When an oscillating system is described by amplitudes ζ or ζ_1 such as those given in Ch. 4, § 2.1 and in Ch. 2, § 4.4, quantities of the forms $\langle\omega|\zeta|^2\rangle$ or $\langle|\dot{\zeta}_1|^2/\omega\rangle$ can be introduced which represent adiabatic invariants.

The definitions of adiabatic invariance can then be stated as in § 2.1, i.e., in the way suggested by CHANDRASEKHAR [1958] and further discussed by VANDERVOORT [1960]. This formulation of the invariance is precise and does not need further comments.

An alternative possibility of defining the invariance arises from the development of the dynamic variables in terms of ε, along the lines of Ch. 3, § 1 and of Ch. 4, § 3. Such a definition has been formulated most in detail by LENARD [1959] who states the following:

(i) Introduce a parameter ε such that $\varepsilon \to 0$ implies the indefinite decrease in the rates of change of external parameters. In other words, this means that the radius of gyration and the gyro time become infinitely small compared to the characteristic changes of the electric and magnetic fields in space and time as ε approaches zero.

(ii) Suppose that a positive constant c_m can be found such that the change $\Delta\chi$ of a certain quantity χ obeys the relation

$$|\Delta\chi| < \varepsilon^{\nu} c_m \qquad (\nu \text{ positive integer}) \tag{4.111}$$

for all sufficiently small ε. Then, χ is said to be an adiabatic invariant to the νth order. This implies that the change $\Delta\chi$ in χ should approach zero at least at the same rate as ε^{ν}.

(iii) If relation (4.111) holds for any ν, the quantity χ is said to be an adiabatic invariant to all orders.

(iv) Suppose, in addition, that $\Delta\chi$ depends on other parameters besides ε. If both the range of ε and the choice of c_m can be made independently of these parameters, χ is said to be an adiabatic invariant uniformly in the parameters.

These statements can be illustrated by the result (4.110) for the magnetic moment M. Since ΔM is at least of the order ε^2 we have $|\Delta M|/\varepsilon^2 =$ const. and M becomes an adiabatic invariant at least to second order.

With KRUSKAL [1957] we should stress here that constancy to all orders does not mean exact constancy, but merely that the deviation from constancy goes to zero faster than any power of ε. As mentioned by KULSRUD [1957] such a situation is illustrated by the function $\chi = \chi_0 + \chi_1 \exp(-1/\varepsilon)$, where χ_0 and χ_1 are constants. Thus, χ is an infinite series in ε which starts with the constant term χ_0. Here $|\Delta\chi|/\varepsilon^{\nu} = |\chi_1 \exp(-1/\varepsilon)|/\varepsilon^{\nu}$ vanishes for any power ν when ε approaches zero. Therefore, χ is an adiabatic invariant to all orders

in ε according to the present definitions. For finite values of ε it is, of course, not a constant.

In most cases of physical interest the Larmor radius and the gyro time are finite as well as ε. Thus, even if a certain quantity is an adiabatic invariant to all orders of the present asymptotic expansions, there is no direct information about its real deviations from constancy. To obtain such an information we have to evaluate the coefficients of the expansions in detail, or we have to use methods such as those demonstrated in § 2.3.

We finally observe that deviations from adiabatic invariance can be produced when the basic assumption (iii) of Ch. 2, § 3.3 is violated. Examples of this are the resonance phenomena between the Larmor motion and the longitudinal oscillations studied by CHIRIKOV [1960]. Such phenomena produce a coupling between these motions, and the latter can no longer be treated as independent degrees of freedom. A study of the deviations requires a special method of analysis.

CHAPTER 5

MACROSCOPIC THEORY

Up to this point the orbits of individual particles have been used to describe the behaviour of an ionized gas. In this chapter the study will continue from a different angle of approach which starts from Boltzmann's equation. Thus, we are going to unite the single particle picture with a macroscopic fluid model. For a large class of problems these approaches will provide equivalent methods of solution.

The equations of motion of this chapter are derived from the moments of the Boltzmann equation and arise from mean values taken over the velocity distribution of particles. Finite Larmor radius effects which are present in the orbit theory will appear also in the macroscopic approach. One important limitation of the latter is that it does not permit a general study of the changes of state in velocity space. Such changes can only be treated in terms of a complete kinetic theory.

For the fluid model to be developed it is necessary to assume that the mean free paths and the mean collision times are short compared to the characteristic macroscopic variations in space and time (GRAD [1961]). The present fluid model is based on such an assumption, but is at the same time restricted to a local thermodynamic state where dissipation effects only have a minor influence on the macroscopic motion.

1. The Boltzmann Equation

A detailed line of deductions which leads to the Boltzmann equation and its moments can be found in several monographs such as those by CHAPMAN and COWLING [1939, 1952], HIRSCHFELDER, CURTISS and BIRD [1954] and BRANDSTATTER [1963]. Only a brief summary will be given here, which does not claim to present a rigorous proof of the obtained results.

1.1. LIOUVILLE'S THEOREM

Consider a gas consisting of N particles the state of which is given by $3N$ pairs (q_k, p_k) of generalized coordinates and momenta. We can then introduce a $6N$-dimensional phase space in which the gas is treated as a single system

with $3N$ degrees of freedom. Its state of motion is given by the position $(q_1, \ldots, q_{3N}, p_1, \ldots, p_{3N})$ of a point in phase space. The corresponding Hamiltonian is $H(q_1, \ldots, q_{3N}, p_1, \ldots, p_{3N})$.

As a next step reproduce the system such as to form a large assembly of identical members, which do not interact with each other. Thus, we fill the $6N$-dimensional phase space with a large number of points at time $t = 0$, each of which represents a system in a certain initial state. We then investigate how this assembly of phase points will move as a function of time. Since the number of systems is fixed, no phase points are created or destroyed. Consequently, we can treat the phase points as a "gas" without sources and sinks and write down the corresponding "equation of continuity". With F as the density of phase points it becomes

$$\frac{\partial F}{\partial t} + \sum_{k=1}^{3N} \left[\frac{\partial}{\partial q_k} (F\dot{q}_k) + \frac{\partial}{\partial p_k} (F\dot{p}_k) \right] = \frac{\partial F}{\partial t} + \{F, H\} = 0 \tag{5.1}$$

when use is made of the canonical equations (2.56) and (2.57), and the Poisson bracket of the second member is defined in analogy with (2.59) for $k = 1, \ldots, 3N$.

Equation (5.1) is *Liouville's theorem*. Since the second member has the form of a total time derivative $\mathrm{d}F/\mathrm{d}t$ of F according to (2.59) we see that the fictious gas of phase points moves as an incompressible fluid in $6N$-dimensional phase space.

Now consider again the physical gas which consists of a very large number N of particles. Make the special assumption that the interactions between the latter are weak enough to be neglected. We can then for a moment consider the particles as N systems without mutual interactions and study their distribution function $f(q_k, p_k, t)$ in the six-dimensional phase space (q_k, p_k). In analogy with (5.1) we then write Liouville's theorem in the form

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{f, H\} = 0. \tag{5.2}$$

1.2. THE VLASOV EQUATION

We shall now make a change of representation from $q_k p_k$ space to $q_k w_k$ space which is more useful in our further treatment of the equations of motion (see e.g. LINHART [1960]). Using rectangular coordinates equations (5.2), (2.61) and (2.62) combine to

$$\frac{\partial f}{\partial t} + w_j \frac{\partial f}{\partial q_j} + \left(q w_k \frac{\partial A_k}{\partial q_j} - q \frac{\partial \phi}{\partial q_j} - m \frac{\partial \phi_g}{\partial q_j}\right) \frac{\partial f}{\partial p_j} = 0. \tag{5.3}$$

The transformation which should be made from (q_k, p_k, t) to (q'_k, w'_k, t') is

$$q_k = q'_k, \quad p_k = m w'_k + q A_k, \quad t = t', \tag{5.4}$$

where

$$\frac{\partial q'_k}{\partial q_j} = \delta_{kj}, \quad \frac{\partial q'_k}{\partial p_j} = 0, \quad \frac{\partial q'_k}{\partial t} = 0, \tag{5.5}$$

$$\frac{\partial w'_k}{\partial q_j} = -\frac{q}{m}\frac{\partial A_k}{\partial q_j}, \quad \frac{\partial w'_k}{\partial p_j} = \frac{\delta_{kj}}{m}, \quad \frac{\partial w'_k}{\partial t} = -\frac{q}{m}\frac{\partial A_k}{\partial t}, \tag{5.6}$$

$$\frac{\partial t'}{\partial q_j} = 0, \quad \frac{\partial t'}{\partial p_j} = 0, \quad \frac{\partial t'}{\partial t} = 1. \tag{5.7}$$

The partial derivatives of f with respect to q_j can now be expanded in the form

$$\frac{\partial f}{\partial q_j} = \frac{\partial f}{\partial q'_k} \cdot \frac{\partial q'_k}{\partial q_j} + \frac{\partial f}{\partial w'_k} \cdot \frac{\partial w'_k}{\partial q_j} + \frac{\partial f}{\partial t'} \cdot \frac{\partial t'}{\partial t} \tag{5.8}$$

and similar relations hold for $\partial f/\partial p_j$ and $\partial f/\partial t$. After insertion of the resulting expressions (5.3) becomes

$$D_w f \equiv \frac{\partial f}{\partial t} + \boldsymbol{w} \cdot \nabla f + \frac{1}{m}(\boldsymbol{F} + q\boldsymbol{w} \times \boldsymbol{B}) \cdot \nabla_w f = 0, \tag{5.9}$$

where the primes have been dropped and $\nabla_w = (\partial/\partial w_x, \partial/\partial w_y, \partial/\partial w_z)$ denotes the gradient in velocity space. In this representation $\boldsymbol{\rho}$, $\boldsymbol{w}$ and t are to be considered as independent variables.

Equation (5.9) is the *Vlasov equation* (or the collisionless Boltzmann equation). It can easily be understood from physical arguments since the acceleration due to the external force $\boldsymbol{F} + q\boldsymbol{w} \times \boldsymbol{B}$ corresponds to an equivalent "velocity" in velocity space and the motion takes place without sources or sinks produced by collisions.

When collisions occur, part of the particles in a volume element of $q_k w_k$ space will be scattered out of the element and another part will enter by scattering from other elements. Equation (5.9) is then modified to the Boltzmann equation

$$D_w f = (\partial f/\partial t)_{\text{coll.}}, \tag{5.10}$$

where the right hand member is the net gain of particles per unit time and

volume of $q_k w_k$ space due to collisions. In a rigorous theory the right hand member of (5.10) can be deduced from the collisional forces in the Hamiltonian of (5.1). The density F is then integrated over the coordinates and momenta of all but one of the particles.

In the next section we shall limit ourselves to a situation where dissipation due to collisions is of minor importance and (5.9) can be used in the derivation of the fluid equations. At the same time we assume collisions to be effective enough to establish a local distribution function of the form $f = f(\boldsymbol{\rho}, \boldsymbol{w}, t)$. These statements are not in contradiction. In terms of a physical picture they imply that collisions in a plasma are frequent enough to establish a local Maxwellian distribution, but are at the same time rare enough for the macroscopic motion to be only slightly affected by dissipation.

1.3. MACROSCOPIC CONSERVATION LAWS

The density $n(\boldsymbol{\rho}, t)$ of a certain kind of particles at a particular point in space and time is

$$n = \iiint_{-\infty}^{+\infty} f(\boldsymbol{\rho}, \boldsymbol{w}, t) \mathrm{d}w_x \,\mathrm{d}w_y \,\mathrm{d}w_z \tag{5.11}$$

with f as the corresponding distribution function. Accordingly, we define the mean value (or moment) of a certain quantity $\chi(\boldsymbol{\rho}, \boldsymbol{w}, t)$ from an average over the velocity distribution:

$$\bar{\chi} = \frac{1}{n(\boldsymbol{\rho}, t)} \iiint_{-\infty}^{+\infty} \chi f \,\mathrm{d}w_x \,\mathrm{d}w_y \,\mathrm{d}w_z. \tag{5.12}$$

Multiply the terms of (5.9) by χ and integrate over velocity space:

$$\iiint_{-\infty}^{+\infty} \chi \frac{\partial f}{\partial t} \mathrm{d}w_x \,\mathrm{d}w_y \,\mathrm{d}w_z = \frac{\partial}{\partial t}(n\bar{\chi}) - n \overline{\frac{\partial \chi}{\partial t}}, \tag{5.13}$$

$$\iiint_{-\infty}^{+\infty} \chi (\boldsymbol{w} \cdot \nabla) f \,\mathrm{d}w_x \,\mathrm{d}w_y \,\mathrm{d}w_z = \operatorname{div} (n\overline{\chi \boldsymbol{w}}) - n\overline{\boldsymbol{w} \cdot \nabla \chi}, \tag{5.14}$$

$$\iiint_{-\infty}^{+\infty} \frac{\chi}{m} (\boldsymbol{F} + q\boldsymbol{w} \times \boldsymbol{B}) \cdot \nabla_w f \,\mathrm{d}w_x \,\mathrm{d}w_y \,\mathrm{d}w_z$$

$$= -(n/m)\overline{\nabla_w \cdot [(\boldsymbol{F} + q\boldsymbol{w} \times \boldsymbol{B})\chi]}. \tag{5.15}$$

Equations (5.13), (5.14) and (5.15) result from partial integration. In (5.15) has been assumed that the product $\chi(\boldsymbol{F} + q\boldsymbol{w} \times \boldsymbol{B})f$ approaches zero when

$|\boldsymbol{w}|$ tends to infinity. This is a justified assumption for distributions f of physical interest. We use these last three equations to obtain

$$\frac{\partial}{\partial t}(n\bar{\chi}) + \nabla \cdot (n\overline{\chi \boldsymbol{w}}) - n\left[\overline{\frac{\partial \chi}{\partial t}} + \overline{\boldsymbol{w} \cdot \nabla \chi} + \frac{1}{m}\overline{(\boldsymbol{F} + q\boldsymbol{w} \times \boldsymbol{B}) \cdot \nabla_w \chi}\right] = 0 \tag{5.16}$$

for the collisionless case, provided that the only velocity dependent part of the total external force is $q\boldsymbol{w} \times \boldsymbol{B}$, the k component of which does not depend on w_k.

The equation of continuity is obtained if we let $\chi = 1$:

$$\frac{\partial n}{\partial t} + \operatorname{div}(n\boldsymbol{v}) = 0, \tag{5.17}$$

where we have defined the mean velocity $\boldsymbol{v} = \overline{\boldsymbol{w}}$ according to equation (5.12).

With $\chi = mw_k$ an equation of motion results which expresses the balance of momentum in the k direction:

$$\frac{\partial}{\partial t}(nm\, v_k) + \operatorname{div}(nm\, \overline{w_k \boldsymbol{w}}) - n(\boldsymbol{F} + q\boldsymbol{v} \times \boldsymbol{B})_k = 0, \tag{5.18}$$

where it is observed that $(\rho, \boldsymbol{w}, t)$ should be considered as independent variables in (5.9) and in the evaluations of expressions (5.13), (5.14) and (5.15). Thus, $\partial w_k/\partial t = 0$ and $(\boldsymbol{w} \cdot \nabla)w_k = 0$ here. Since $\boldsymbol{F}$ and $\boldsymbol{B}$ do not depend on $\boldsymbol{w}$ we have $\boldsymbol{F} = \overline{\boldsymbol{F}}$ and $\boldsymbol{B} = \overline{\boldsymbol{B}}$.

Introduce the deviation $\tilde{\boldsymbol{w}} = \boldsymbol{w} - \boldsymbol{v}$ of the velocity $\boldsymbol{w}$ from its average value. Accordingly $\overline{\tilde{\boldsymbol{w}}} = 0$ and we can define a *pressure tensor*

$$\pi = (\pi_{jk}), \quad \pi_{jk} = nm\, \overline{\tilde{w}_j \tilde{w}_k}, \quad \tilde{\boldsymbol{w}} = \boldsymbol{w} - \boldsymbol{v}. \tag{5.19}$$

Equation (5.18) can now be rewritten as

$$nm\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}\right] = n(\boldsymbol{F} + q\boldsymbol{v} \times \boldsymbol{B}) - \operatorname{div} \pi \tag{5.20}$$

when use is made of the equation (5.17) of continuity.

A special case of interest is that where the off-diagonal elements of the pressure tensor can be neglected in a coordinate system having one axis along $\boldsymbol{B}$. The pressure is assumed to be anisotropic with the components $p_{\parallel}$ and $p_{\perp}$ parallel with and perpendicular to the magnetic field lines. With

ERRATA

Page 112, 10th line from the bottom, *read*: $\overline{\tilde{w}}$ *instead of* $\overline{\tilde{w}}'$.
Page 122, 6th line from the bottom, *read*: d/dt *instead of* dv/dt.
Page 136, legend of Figure 6.3, 1st line, *read*: s_1 and s_2 *instead of* s_1 and s.
Page 140, 7th line from the bottom, *read*: $(\varkappa_\perp)^{-3/2}$ *instead of* $(\varkappa_\perp)^{-\frac{3}{2}}$.
Page 147, 10th line, *read*: $\dfrac{\partial}{\partial t}$ *instead of* $\dfrac{1}{\partial t}$.
Page 147, 13th line from the bottom, *read*: V_A^2 *instead of* V_A.
Page 167, 14th line from the bottom, *read*: by BONNEVIER and LEHNERT [1960] *instead of* BONNEVIER and LEHNERT [1960] by.
Page 179, 16th line from the bottom, *read*: enhance *instead of* enhances.
Page 186, 10th line from the bottom, *read*: have obtained *instead of* obtained.
Page 188, 5th line, *read*: change *instead of* changes.
Page 205, 11th line, *read*: $\alpha_{1e} = k_x g/\omega_1$ *instead of* $\alpha_{1e} = k_x g/u_i$.
Page 252, 6th line from the bottom, *read*: $(\frac{1}{2}\mu_r)^{3/2}$ *instead of* $(\frac{1}{2}\mu_r)^{3/2}$.
Page 264, 6th line from the bottom, *read*: y_1 *instead of* v_1.

CHEW *et al.* [1956] and CHANDRASEKHAR *et al.* [1957] we can then write the pressure tensor in terms of the unit vector $\hat{\boldsymbol{B}} = \boldsymbol{B}/B$ along the magnetic field:

$$\pi_{jk} = p_{\|}\hat{B}_j\hat{B}_k + p_{\perp}(\delta_{jk} - \hat{B}_j\hat{B}_k). \tag{5.21}$$

The divergence of this tensor becomes

$$\begin{aligned}\operatorname{div} \pi = \hat{\boldsymbol{B}}\,[(\hat{\boldsymbol{B}}\cdot\nabla)p_{\|} + (p_{\|} - p_{\perp})\operatorname{div}\hat{\boldsymbol{B}}]\\ + [\nabla - \hat{\boldsymbol{B}}(\hat{\boldsymbol{B}}\cdot\nabla)]p_{\perp} + (p_{\|} - p_{\perp})(\hat{\boldsymbol{B}}\cdot\nabla)\hat{\boldsymbol{B}}.\end{aligned} \tag{5.22}$$

Using the identity $B \operatorname{div} \hat{\boldsymbol{B}} = -(\hat{\boldsymbol{B}}\cdot\nabla)\, B$ and the parallel and transverse gradients

$$\nabla_{\|} = \hat{\boldsymbol{B}}(\hat{\boldsymbol{B}}\cdot\nabla), \quad \nabla_{\perp} = \nabla - \nabla_{\|} \tag{5.23}$$

the divergence becomes

$$\operatorname{div} \pi = \nabla_{\|}\, p_{\|} - \frac{p_{\|} - p_{\perp}}{B}\nabla_{\|} B + \nabla_{\perp}p_{\perp} + (p_{\|} - p_{\perp})(\hat{\boldsymbol{B}}\cdot\nabla)\hat{\boldsymbol{B}}. \tag{5.24}$$

The first two terms of this equation are directed along $\boldsymbol{B}$ and the last two terms are perpendicular to $\boldsymbol{B}$, as can be seen from equation (3.21). Especially for isotropic pressure distributions $p_{\|} = p_{\perp}$ and

$$\operatorname{div} \pi = \nabla p, \tag{5.25}$$

where p is the scalar pressure.

Next turn to the equations expressing conservation of energy. With $\chi = \frac{1}{2}mw_k^2$ the general equation (5.16) obtains the form

$$\begin{aligned}\tfrac{1}{2}\frac{\partial}{\partial t}(nmv_k^2) + \tfrac{1}{2}\operatorname{div}(nmv_k^2\boldsymbol{v}) + \frac{\partial}{\partial t}(nm\, U_{(k)}) + \operatorname{div}(nm\, U_{(k)}\boldsymbol{v}) + \operatorname{div}\boldsymbol{Q}_{(k)}\\ + \operatorname{div}(nmv_k\,\overline{\tilde{w}_k\tilde{\boldsymbol{w}}}) - n(\boldsymbol{F} + q\boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{v}_k - nq\overline{(\tilde{\boldsymbol{w}}\times\boldsymbol{B})\cdot\tilde{\boldsymbol{w}}_k} = 0,\end{aligned} \tag{5.26}$$

where we have introduced the thermal energy $U_{(k)} = \frac{1}{2}\,\overline{\tilde{w}_k^2}$ per unit mass and the heat flow vector

$$\boldsymbol{Q}_{(k)} = \tfrac{1}{2}\, nm\, \overline{\tilde{w}_k^2\tilde{\boldsymbol{w}}} \tag{5.27}$$

associated with the velocity $\boldsymbol{w}_k = w_k\,\hat{\boldsymbol{k}}$ in the k direction. The sixth term of (5.26) can be written as

$$\operatorname{div}(nmv_k\,\overline{\tilde{w}_k\tilde{\boldsymbol{w}}}) = \operatorname{div}(\boldsymbol{v}_k\cdot\pi) - \boldsymbol{v}_k\operatorname{div}\pi \equiv \pi : \nabla\boldsymbol{v}_k, \tag{5.28}$$

where $\pi : \nabla$ stands for the product between the pressure tensor (5.19) and the operator ∇. Multiply (5.18) by v_k and subtract the obtained result from (5.26). After combination with the equation (5.17) of continuity follows that

$$\frac{\partial}{\partial t}(nmU_{(k)}) + \operatorname{div}(nmU_{(k)}\boldsymbol{v}) + \operatorname{div}\boldsymbol{Q}_{(k)} + \pi : \nabla \boldsymbol{v}_k - nq\,\overline{(\tilde{\boldsymbol{w}}_k \times \tilde{\boldsymbol{w}})} \cdot \boldsymbol{B} = 0 \tag{5.29}$$

which expresses conservation of energy for the part $U_{(k)}$. For the total kinetic energy $U_{(\mathrm{tr})} = \frac{1}{2}\overline{\tilde{w}^2}$ per unit mass we have from (5.29)

$$\frac{\partial}{\partial t}(nmU_{(\mathrm{tr})}) + \operatorname{div}(nmU_{(\mathrm{tr})}\boldsymbol{v}) + \operatorname{div}\boldsymbol{Q} + \pi : \nabla \boldsymbol{v} = 0 \tag{5.30}$$

with the total heat flow vector

$$\boldsymbol{Q} = \tfrac{1}{2}\, nm\, \overline{\tilde{w}^2 \tilde{\boldsymbol{w}}}. \tag{5.31}$$

In the special case where the pressure tensor is given by the two scalar quantities $p_{\parallel}$ and $p_{\perp}$ as shown in (5.21) we have

$$\pi : \nabla \boldsymbol{v}_k = (p_{\parallel} - p_{\perp})\hat{\boldsymbol{B}}(\hat{\boldsymbol{B}} \cdot \nabla)\boldsymbol{v}_k + p_{\perp} \operatorname{div} \boldsymbol{v}_k. \tag{5.32}$$

Further, assume adiabatic changes of state where the heat flow can be neglected. Then, the equation (5.30) expressing conservation of the total thermal energy is easily shown to become

$$\tfrac{1}{2}\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla\right)(p_{\parallel} + 2p_{\perp}) + (p_{\parallel} - p_{\perp})\hat{\boldsymbol{B}}(\hat{\boldsymbol{B}} \cdot \nabla)\boldsymbol{v} + \tfrac{1}{2}(p_{\parallel} + 4p_{\perp}) \operatorname{div} \boldsymbol{v} = 0. \tag{5.33}$$

On the other hand, using (5.29) with $\boldsymbol{w}_k = \boldsymbol{v}_{\parallel} = \hat{\boldsymbol{B}} \cdot v_{\parallel}$ and observing that $\hat{\boldsymbol{B}}(\hat{\boldsymbol{B}} \cdot \nabla)\boldsymbol{v}_{\parallel} = \operatorname{div} \boldsymbol{v}_{\parallel}$ we obtain for the longitudinal direction

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla\right)p_{\parallel} = -\, 2p_{\parallel}\hat{\boldsymbol{B}}(\hat{\boldsymbol{B}} \cdot \nabla)\boldsymbol{v} - p_{\parallel} \operatorname{div} \boldsymbol{v} \tag{5.34}$$

which is also obvious from (5.33) when $p_{\parallel}$ and $p_{\perp}$ are independent of each other. The difference between equations (5.33) and (5.34) gives an energy equation for the transverse direction:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla\right)p_{\perp} = p_{\perp}\hat{\boldsymbol{B}}(\hat{\boldsymbol{B}} \cdot \nabla)\boldsymbol{v} - 2p_{\perp} \operatorname{div} \boldsymbol{v}. \tag{5.35}$$

The adiabatic relations (5.34) and (5.35) were first deduced by CHEW *et al.* [1956].

CHAPMAN and COWLING [1939], MARSHALL [1958] and KAUFMAN [1960] have included the effects of viscosity in the macroscopic equations and the latter author has interpreted his results in terms of the single particle motion. When the self-collision frequency between like particles is smaller than the gyro frequency a highly anisotropic situation arises. As a consequence the shear forces are cut down appreciably in a plane perpendicular to a strong magnetic field.

The present deductions have been carried out under the assumption that dissipation effects due to collisions are very small and do not give any essential contribution to equation (5.16). We still have to assume that the mean free paths and the mean collision times are short compared to the characteristic variations of our fluid model in space and time. If these conditions cannot be satisfied the latter usually becomes inapplicable, and the distribution function f then obtains a complex form. Only in some special cases, such as when conditions are uniform along a magnetic field line, a fluid model may still be applicable to the motion across the lines of force (SPITZER [1960], GRAD [1961]).

*1.4. APPROXIMATE SOLUTIONS OF THE VLASOV EQUATION

The Vlasov equation (5.9) can be solved by sucessive approximations of an expansion, which is developed in terms of the "smallness" parameter ε. Our basic assumptions will become the same as in Ch. 3, § 1, i.e., the magnetic field and the applied force field should vary slowly as prescribed by conditions (3.1), (3.2) and (3.34). A detailed treatment of the problem is beyond the scope of this volume. We only summarize some of the results obtained by MARSHALL [1958] and THOMPSON [1961].

With $\boldsymbol{F} = q\boldsymbol{E}$ we rewrite the Vlasov equation as

$$-(\boldsymbol{E}_\perp + \boldsymbol{w} \times \boldsymbol{B}) \cdot \nabla_w f = \varepsilon \left[\frac{\partial f}{\partial t} + (\boldsymbol{w} \cdot \nabla) f + \frac{E_\parallel}{\varepsilon} \frac{\partial f}{\partial w_\parallel} \right], \quad \varepsilon = \frac{m}{q} \tag{5.36}$$

and observe that $E_\parallel/\varepsilon$ should be of zero or higher order according to equation (3.34). We now make a transformation in velocity space and change the independent variables from $(\rho, \boldsymbol{w}, t)$ to $(\rho, \boldsymbol{w}', t)$ where the velocity $\boldsymbol{w}'$ is defined by

$$\boldsymbol{w}' = \boldsymbol{w} - \boldsymbol{u}', \tag{5.37}$$

and

$$\boldsymbol{E}_\perp + \boldsymbol{u}' \times \boldsymbol{B} = 0. \tag{5.38}$$

To lowest order $\boldsymbol{w}'_\perp$ becomes equal to the velocity $\boldsymbol{W}$ of gyration.

Finally introduce a polar representation of the velocity $\boldsymbol{w}'_\perp$ where

$$\boldsymbol{w}'_\perp = w'_\perp (\cos\varphi, \sin\varphi). \tag{5.39}$$

The result of these transformations is that (5.36) obtains the form

$$\frac{\partial f}{\partial \varphi} = \frac{\varepsilon}{B}\,\mathrm{D}f. \tag{5.40}$$

Here the operator D is defined by

$$\mathrm{D}f = \mathrm{D}'f - (\mathrm{D}'\boldsymbol{u}')\cdot\frac{\partial f}{\partial \boldsymbol{w}'} - \boldsymbol{w}'\cdot(\mathrm{D}'\hat{\boldsymbol{B}})\frac{\partial f}{\partial w'_\parallel} + w'_\parallel(\mathrm{D}'\hat{\boldsymbol{B}})\frac{\partial f}{\partial \boldsymbol{w}'} + \frac{E_\parallel}{\varepsilon q}\,\frac{\partial f}{\partial w'_\parallel}$$

with (5.41)

$$\mathrm{D}' = \frac{\partial}{\partial t} + (\boldsymbol{w}' + \boldsymbol{u}')\cdot\nabla \tag{5.42}$$

and $\partial/\partial\boldsymbol{w}' \equiv \nabla'_w$.

The right hand member of (5.40) is of order ε. Thus, the zero order solution f_0 of the equation is obtained when this member is dropped. The corresponding solution has the general form

$$f_0 = f_0(w'^2_\perp, w'_\parallel, \rho, t) \tag{5.43}$$

which is independent of φ. Remembering that ε represents the ratio $1/\omega_g t_c \ll 1$ when equation (5.40) is written in dimensionless form we now expand f in powers of ε:

$$f = \sum_\nu \varepsilon^\nu f^{(\nu)} \quad (\nu = 0, 1, \ldots). \tag{5.44}$$

With this expansion inserted into (5.40) we obtain a relation which should be valid for any value of ε and therefore has to be satisfied for every separate power of ε. Consequently,

$$\frac{\partial f^{(\nu)}}{\partial\varphi} = \left[\frac{1}{\omega_g(\rho, t)}\right]\mathrm{D}f^{(\nu-1)}, \quad (\nu = 1, 2, \ldots). \tag{5.45}$$

The first order solution is now obtained from equations (5.43) – (5.45) and becomes $f_1 = f^{(0)} + \varepsilon f^{(1)}$, where $f^{(0)} = f_0$ and

$$f^{(1)} = g_1(w'^2_\perp, w'_\parallel, \rho, t) + \frac{1}{\omega_g}\int \mathrm{D}f_0\,\mathrm{d}\varphi. \tag{5.46}$$

We must seek solutions which are periodic in φ and a constraint is therefore placed upon f_0, namely

$$\int_0^{2\pi} \mathrm{D}f_0\,\mathrm{d}\varphi = 0. \tag{5.47}$$

This relation can be used to rewrite the inhomogeneous part of f_1 given by the last term of equation (5.46). At each stage in this solution procedure a similar constraint must be applied.

We shall not write down the resulting expression for f_1 in detail. It should only be mentioned that f_1 can be used to calculate the components of the pressure tensor by forming the moments with the velocity components as given by equations (5.19) and (5.12). With the z axis along a nearly homogeneous field $\boldsymbol{B}$ and for a velocity field situated in the xy plane

$$\pi_{xx} = p_\perp - \tfrac{1}{4} nm\, \omega_g \bar{a}^2 \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right), \tag{5.48}$$

$$\pi_{yy} = p_\perp + \tfrac{1}{4} nm\, \omega_g \bar{a}^2 \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right), \tag{5.49}$$

$$\pi_{xy} = \pi_{yx} = \tfrac{1}{4} nm\, \omega_g \bar{a}^2 \left(\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y} \right), \tag{5.50}$$

as given by THOMPSON [1961] (see also ROBERTS and TAYLOR [1962]). In equations (5.48) – (5.50) $p_\perp$ is the transverse scalar pressure in zero order and a is the Larmor radius. To lowest order the transverse macroscopic fluid velocity $\boldsymbol{v}_\perp$ in equations (5.48) – (5.50) can be substituted by the transverse electric drift $\boldsymbol{u}_E$.

2. Connexion between Microscopic and Macroscopic Theories

The orbit theory of Chapter 3 and the present macroscopic treatment are both based on the collisionless equation of motion of individual particles. They therefore constitute equivalent methods of approach. In this paragraph we shall try to establish a closer connexion between the two methods and clarify some of the apparent contradictions which arise when the guiding centre approach is compared with the macroscopic fluid equations. Among the discussions on this subject may be mentioned those by SCHLÜTER [1952], SPITZER [1952, 1956], LONGMIRE [1963], CHANDRASEKHAR *et al.* [1958a], ÅSTRÖM [1958], NORTHROP [1960] and LUNDQUIST [1960].

2.1. THE EQUATION OF MOTION

We shall connect the equation of motion (5.20) with the relations (3.47) and (3.49) deduced from the orbit theory. For this purpose divide (5.20)

into its longitudinal and transverse parts,

$$nm\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right]_{\parallel} = n\boldsymbol{F}_{\parallel} - (\operatorname{div} \pi_{\parallel}) \tag{5.51}$$

and

$$n\boldsymbol{v}_{\perp} = (n\boldsymbol{F} - \operatorname{div}\pi) \times \boldsymbol{B}/qB^2 - nm\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] \times \boldsymbol{B}/qB^2, \tag{5.52}$$

where the last equation has been obtained from the vector product of (5.20) with $\boldsymbol{B}$. The limit $\boldsymbol{B} = 0$ is excluded. Observe that the derivative, $\partial/\partial t + \boldsymbol{v}\cdot\nabla$, following the mean mass motion $\boldsymbol{v}$ is *not* necessarily equal to the total time derivative, $\partial/\partial t + \boldsymbol{u}\cdot\nabla$, following the drift of individual particles; there is a difference because $\boldsymbol{v}$ and $\boldsymbol{u}$ are generally not equal.

We restrict the present discussion to a pressure tensor of the form (5.21), which can be described by two scalar pressures $p_{\parallel} = 2nK_{\parallel}$ and $p_{\perp} = nK_{\perp}$. Here $K_{\parallel}$ and $K_{\perp}$ denote the corresponding thermal energies as defined in (3.45). Assume that the mean velocity $\overline{\boldsymbol{u}}_{\parallel}$ is small compared to the thermal velocity $(\overline{\tilde{\boldsymbol{u}}_{\parallel}{}^2})^{\frac{1}{2}}$. By means of (5.24) we can immediately rewrite equation (5.52) in the form

$$\begin{aligned} n\boldsymbol{v}_{\perp} &= n\boldsymbol{F} \times \boldsymbol{B}/qB^2 + n(K_{\perp} - 2K_{\parallel})(\nabla B - \hat{\boldsymbol{B}} \times \operatorname{curl}\boldsymbol{B}) \times \boldsymbol{B}/qB^3 \\ &\quad - \nabla(nK_{\perp}) \times \boldsymbol{B}/qB^2 - nm\frac{\overline{\mathrm{d}\boldsymbol{u}_{\perp}}}{\mathrm{d}t} \times \boldsymbol{B}/qB^2 \\ &\quad - nm\left[\frac{\partial \boldsymbol{v}_{g}}{\partial t} + (\boldsymbol{v}_{g}\cdot\nabla)(\boldsymbol{v}_{g} + \overline{\boldsymbol{u}}) + (\overline{\boldsymbol{u}}\cdot\nabla)\boldsymbol{v}_{g}\right] \times \boldsymbol{B}/qB^2. \end{aligned} \tag{5.53}$$

Here we have introduced the velocity

$$\boldsymbol{v}_{g} = \boldsymbol{v} - \overline{\boldsymbol{u}} = \boldsymbol{v} - \overline{\boldsymbol{u}}_{\parallel} - \overline{\boldsymbol{u}}_{\perp} \tag{5.54}$$

which represents the particular part of the total particle flux $n\boldsymbol{v}$ per unit area which is due to the gyration $\boldsymbol{W}$ only, and not due to the mean flux $n\overline{\boldsymbol{u}}$ of the guiding centra. To obtain (5.53) we have also used relations (3.19), (3.20) and (3.21) to rewrite $\overline{\mathrm{d}\boldsymbol{u}/\mathrm{d}t}$. An acceleration term, $\overline{u_{\parallel}\mathrm{d}\hat{\boldsymbol{u}}_{\parallel}/\mathrm{d}t}$, originates from the curvature of the magnetic field lines. It gives rise to the factor 2 in front of $K_{\parallel}$ in equation (5.53). Contributions from the last term of (5.53) might appear to be negligible compared to other terms in the same equation. Nevertheless erraneous results will be obtained in certain cases if the contributions from $\boldsymbol{v}_{g}$ are discarded. Examples of this will be given in Ch. 8, §§ 2.4 and 2.5.

Comparing (3.47) deduced from the first order orbit theory with (5.53) we see that there is complete agreement in the zero order approximation when the thermal motion disappears and $K_{\parallel} = 0$, $K_{\perp} = 0$, $\boldsymbol{v}_g = 0$. Of course, the result requires the mean longitudinal drift $\overline{u}_{\parallel}$ to be of the same order as $\boldsymbol{v}_{\perp}$ and $\boldsymbol{v}$. This is also consistent with the last of equations (3.34).

Under certain conditions there is also agreement between equations (3.47) and (5.53) in first order. First observe that the acceleration term of (5.52) is at least of first order in ε compared to the left hand member of the same equation. We further notice that the ratio between the div π and $n\boldsymbol{F}$ terms in (5.52) is of the order of $kT_c/L_{c\pi}F_c$, where T_c, $L_{c\pi}$ and F_c are characteristic values of the temperature, the spatial variations of the pressure tensor and of the force field, respectively. This ratio sometimes becomes small under conditions of physical interest. This occurs e.g. for $\boldsymbol{F} = q\boldsymbol{E} = -q\nabla\phi$ when the electric potential difference associated with the Larmor energy is small compared to the potential difference arising from $\boldsymbol{E}$ across the characteristic length of the pressure distribution. The lowest order forms of equations (5.52) and (5.54) therefore become

$$\boldsymbol{v} \approx \boldsymbol{v}_{\parallel} + \boldsymbol{F} \times \boldsymbol{B}/qB^2, \quad \boldsymbol{v}_{\parallel} \approx \overline{\boldsymbol{u}}_{\parallel}, \quad v_g = \mathrm{O}(\varepsilon), \tag{5.55}$$

but this result holds only when div π is small compared to $n\boldsymbol{F}$. Observe that it is not applicable when the thermal velocity by far exceeds that of the macroscopic motion and when the pressure tensor gives contributions to the total particle flux which are of the same importance as those arising from the guiding centre drift. In such a case $\boldsymbol{v}_g$ cannot be treated as a first order quantity. Thus, there will be an agreement in first order between equations (3.47) and (5.53) only when $\boldsymbol{v}_g$ is of first or higher order in ε compared to $\boldsymbol{v}_{\perp}$.

Higher order approximations cannot be discussed here since equation (3.47) has been deduced only in terms of a first order theory. We have already seen in Ch. 3, § 2 that the flux of guiding centra is not necessarily equal to the flux of particles, and we shall discuss this further in § 2.3 of this chapter. In fact, it should also be expected that the inertia force associated with the guiding centre motion is not necessarily equal to the rate of change of the momentum stored in a volume element of an ionized gas. This rate is governed not only by the acceleration of the guiding centra but also by the momentum changes of the corresponding Larmor motion. The latter are contained in the last term of (5.53). To derive more accurate expressions for the dynamics of ionized matter we have to evaluate higher order approximations, not only for the guiding centre drift but also for the Larmor motion.

This is readily illustrated by the examples of Ch. 8, §§ 2.4 and 2.5 on the effects of a density gradient and a finite Larmor radius. There it will be shown that contributions from the pressure tensor due to the last terms of (5.53) and (5.48) – (5.50) are not always negligible in a hot plasma.

We next turn to the longitudinal direction. With the definitions $p_{\parallel} = 2nK_{\parallel}$ and $p_{\perp} = nK_{\perp}$ and the longitudinal component of div $\boldsymbol{\pi}$ given by (5.24) it is immediately seen that (5.51) is consistent with the result (3.49) of the orbit theory.

Consequently, the single particle picture and the macroscopic fluid model yield equations of motion which agree within the range of the approximations underlying the first order orbit theory.

We have just seen that the equations of motion are obtained quite easily from a macroscopic approach whereas the corresponding results can be deduced from the orbit theory first after a number of approximations and by means of rather laborious methods. However, this does not mean that a macroscopic treatment is free from difficulties. In fact, problems arise which mainly concern the determination of the pressure tensor. As shown in § 1.4 the form of the latter is modified in higher orders where it cannot be represented by simple scalar pressures.

2.2. ADIABATIC CHANGES OF STATE

Conservation of momentum was treated in the preceding paragraph both in terms of orbit theory and of macroscopic theory. Here we shall establish a corresponding connexion between the same theories for the conservation of energy.

We start with the law (2.1) of electromagnetic induction. Restricting the present discussion to the lowest order forms of equations (5.20) and (5.52) and to the conditions underlying (5.55) we have

$$\boldsymbol{F}/q = \boldsymbol{E} - (m/q)\nabla\phi_g = -\,\boldsymbol{v} \times \boldsymbol{B}. \tag{5.56}$$

The induction law (2.1) therefore yields

$$\frac{\partial \boldsymbol{B}}{\partial t} = \operatorname{curl}(\boldsymbol{v} \times \boldsymbol{B}) = -\,(\boldsymbol{v} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{v} - \boldsymbol{B} \operatorname{div} \boldsymbol{v}. \tag{5.57}$$

Assume the pressure tensor to be given by (5.21) and substitute $(\boldsymbol{B} \cdot \nabla)\boldsymbol{v}$ from (5.57) into equations (5.34) and (5.35). With div $\boldsymbol{v}$ expressed by (5.17)

the result becomes

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla\right)\left(\frac{p_{\parallel} B^2}{n^3}\right) = 0 \tag{5.58}$$

and

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla\right)\left(\frac{p_{\perp}}{nB}\right) = 0. \tag{5.59}$$

These equations, which represent conservation of energy for the longitudinal and transverse thermal motions, have first been derived by CHEW *et al.* [1956]. In the isotropic case where $p_{\parallel} = p_{\perp} = p$ they should be substituted by the corresponding result derived from equations (5.33) and (5.17), i.e., by

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla\right)\left(\frac{p}{n^{5/3}}\right) = 0 \tag{5.60}$$

which is the well-known adiabatic law.

In zero order the mass velocity $\boldsymbol{v}$ has the solution (5.55) which is equal to the zero order mean value of the guiding centre drift $\boldsymbol{u}$. According to equations (5.59) and (5.58) the quantities

$$\frac{p_{\perp}}{nB} = \frac{\overline{\frac{1}{2}mW^2}}{B} = \bar{M}, \quad \frac{p_{\parallel} B^2}{n^3} = \frac{\overline{m\ddot{u}_{\parallel}^2 B^2}}{n^2} \equiv C_{\parallel} \tag{5.61}$$

then become adiabatic invariants in a frame of reference which follows the mean drift of the particles.

The first of the invariants, $\bar{M}$, is the mean of the equivalent magnetic moment. By means of a rough consideration the second invariant, $C_{\parallel}$, is easily seen to be connected with the longitudinal invariant J of Chapter 4. Suppose that a constant number of particles of mean density n is enclosed in a tube of magnetic flux Φ and mean area $S = \Phi/B$, somewhat like that shown in Figure 4.4. The mean distance of the magnetic mirror points between which the particles are reflected is $2s_{\mathrm{m}}$. Since the total number of particles is conserved $nSs_{\mathrm{m}} = \Phi(n/B)s_{\mathrm{m}}$ must be so, too. Combination with the second invariant of (5.61) then shows that $\overline{\tilde{u}_{\parallel}^2 s_{\mathrm{m}}^2}$ must be constant. This latter quantity has the form of J^2 as seen from the definition (4.8) of the longitudinal invariant.

Finally (5.56) immediately shows that the velocity field $\boldsymbol{v}$ satisfies condition (2.29) for flux-preservation in zero order. Consider an axially symmetric configuration like that of Figure 4.1c. Suppose a mass velocity $\boldsymbol{v}$ to be

produced by an axially symmetric force field $\boldsymbol{F}$. In zero order $\boldsymbol{v}$ then becomes equal to the velocity $\boldsymbol{u}$ of the guiding centra. Further study a curve C which is carried along by the velocity field $\boldsymbol{v}$. It should have the form of a coaxial ring which coincides at a certain time t_0 with the particles at a certain distance from the axis. At all later times the curve will then be situated on the same field lines as the particles with which it coincided at time t_0. It therefore encloses the constant flux Φ of a time-dependent magnetic field. This implies that Φ is an invariant in zero order, in the same sense as in Ch. 4, § 1.5.

The present discussions on flux preservation and on the adiabatic invariants of (5.61) are connected with the magnetic compression phenomena of Chapter 6.

2.3. FLOW OF PARTICLES, GUIDING CENTRA AND DENSITY DISTRIBUTIONS

So far equations of motion have been established for the drift velocity $\boldsymbol{u}$ associated with the flux of guiding centra and for the macroscopic fluid velocity (mass velocity) $\boldsymbol{v}$ associated with the flux of particles. We shall now examine the physical reasons for these fluxes to differ. In this connexion we will also find it useful to study how the surfaces of constant density are moving, i.e., in which way a drift of density distributions will develop. This is of particular interest in problems where charge separation occurs from ion and electron distributions which move at different velocities.

A divergence operation on (5.52) yields in combination with the equation (5.17) of continuity (cf. LEHNERT [1962a]):

$$\begin{aligned}
-\frac{\partial n}{\partial t} &= \left(\boldsymbol{F} \times \boldsymbol{B}/qB^2 - m\frac{\mathrm{d}_v \boldsymbol{v}}{\mathrm{d}t} \times \boldsymbol{B}/qB^2 + \boldsymbol{v}_{\parallel}\right) \cdot \nabla n \\
&\quad + (2/qB^3)(\boldsymbol{B} \times \nabla B) \cdot \operatorname{div} \pi + n \operatorname{div}\left(\boldsymbol{F} \times \boldsymbol{B}/qB^2 - m\frac{\mathrm{d}_v \boldsymbol{v}}{\mathrm{d}t} \times \boldsymbol{B}/qB^2\right) \\
&\quad + (1/qB^2)\operatorname{div}(\boldsymbol{B} \times \operatorname{div} \pi) + nB(\hat{\boldsymbol{v}}_{\parallel} \cdot \nabla)(v_{\parallel}/B),
\end{aligned} \tag{5.62}$$

where we have introduced the derivative $\mathrm{d}_v/\mathrm{d}t \equiv \partial/\partial t + \boldsymbol{v} \cdot \nabla$ following the mass motion and div $\boldsymbol{v}_{\parallel}$ has been rewritten by means of the condition $\operatorname{div} \boldsymbol{B} = 0$.

First study (5.62) in zero order by dropping pressure and inertia terms and assuming the conditions of (5.55) to be applicable:

$$\frac{\partial n}{\partial t} + (\boldsymbol{v}_{\parallel} + \boldsymbol{u}_F) \cdot \nabla n + n \operatorname{div} \boldsymbol{u}_F + nB(\hat{\boldsymbol{v}}_{\parallel} \cdot \nabla)(v_{\parallel}/B) = 0, \tag{5.63}$$

where the external force drift $\boldsymbol{u}_F$ has been defined in equation (3.23). Especially when $v_\parallel/B$ is constant the flow along the magnetic field lines is incompressible and does not give rise to any density changes. Further, assume the force field $\boldsymbol{F}$ to arise from a scalar potential and the magnetic field to be near to a vacuum field, which varies slowly in space compared to the density distribution. This implies that we can put div $\boldsymbol{u}_F = 0$. Equation (5.63) then shows that the density distribution n is displaced with undistorted form at the zero order velocity $\boldsymbol{v}_\parallel + \boldsymbol{u}_F$ of the guiding centra and the mass motion. This is also obvious from a divergence operation on equation (3.43).

Next take first order effects into account and assume the pressure tensor to be given by equations (5.21) and (5.24). We then have

$$(\boldsymbol{B}\times\nabla B)\cdot \operatorname{div}\boldsymbol{\pi} = (\boldsymbol{B}\times\nabla B)\cdot\nabla_\perp p_\perp - (p_\parallel - p_\perp)(\boldsymbol{B}\times\nabla B)\cdot(\boldsymbol{B}\times\operatorname{curl}\boldsymbol{B})/B^2 \tag{5.64}$$

when use is made of (3.21). Further

$$\boldsymbol{B}\times\operatorname{div}\boldsymbol{\pi} = \boldsymbol{B}\times\nabla_\perp p_\perp + (p_\parallel - p_\perp)\boldsymbol{B}\times[(\hat{\boldsymbol{B}}\cdot\nabla)\hat{\boldsymbol{B}}]. \tag{5.65}$$

Introduce the velocity

$$\boldsymbol{U}_n = \boldsymbol{v}_\parallel + \boldsymbol{u}_F + (K_\perp + 2K_\parallel)\boldsymbol{B}\times\nabla B/qB^3 + 2K_\parallel(\operatorname{curl}\boldsymbol{B})_\perp/qB^2 - m\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}_\parallel + \boldsymbol{u}_F)\times\boldsymbol{B}/qB^2 \tag{5.66}$$

and (5.62) reduces in first order to

$$\begin{aligned}\frac{\partial n}{\partial t} &+ \boldsymbol{U}_n\cdot\nabla n + n\operatorname{div}\left[\boldsymbol{u}_F + \frac{m}{qB^2}\boldsymbol{B}\times\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}_\parallel + \boldsymbol{u}_F)\right]\\ &+ \frac{n}{qB^2}[2\hat{\boldsymbol{B}}\times\nabla B + (\operatorname{curl}\boldsymbol{B})_\perp]\cdot\nabla K_\perp\\ &+ \frac{n}{qB^2}\operatorname{div}\{(2K_\parallel - K_\perp)[\hat{\boldsymbol{B}}\times\nabla B + (\operatorname{curl}\boldsymbol{B})_\perp]\}\\ &- \frac{2n}{qB^5}(2K_\parallel - K_\perp)(\boldsymbol{B}\times\nabla B)\cdot(\boldsymbol{B}\times\operatorname{curl}\boldsymbol{B}) + nB(\hat{\boldsymbol{v}}_\parallel\cdot\nabla)\left(\frac{v_\parallel}{B}\right) = 0.\end{aligned} \tag{5.67}$$

Here (3.21) has been applied for a second time and $p_\parallel = 2nK_\parallel$, $p_\perp = nK_\perp$.

In (5.67) all first order effects are included which determine the motion of the surfaces of constant density. We first consider the velocity $\boldsymbol{U}_n$ which gives rise to a convection of these surfaces. If only the two first terms of

(5.67) would be present the density distributions would be displaced with undistorted forms at the velocity $\boldsymbol{U}_n$. Observe that the latter contains all the guiding centre drifts deduced in Ch. 3, § 1.1, including those which arise from the magnetic field gradient. Thus, the contribution to $\boldsymbol{U}_n$ in the transverse direction is equal to the mean drift $\bar{\boldsymbol{u}}_\perp$ given by (3.22) in first order, when an average is taken over the velocity distribution.

We further observe that the magnetic gradient drift (3.24) depends upon the velocities $u_\parallel$ and W of a particle. Thus, in absence of collisions different parts of the particle distribution in velocity space will drift at different speeds. The velocity spectrum will then change in space and time on account of the inhomogeneity of the magnetic field. However, in this simplified analysis we introduce collisions as stated at the beginning of this chapter, and assume a local thermal equilibrium to develop. The entire distribution of particles will then tend to move in space at a speed given by the thermal mean value of the drift velocity. This is shown by the third and fourth terms of the expression (5.66) for $\boldsymbol{U}_n$.

The fourth and fifth terms of (5.67) contain the gradients of the mean thermal energies, $K_\parallel$ and $K_\perp$, and give rise to a distortion of the density distribution when the latter is observed in a coordinate system which moves with the convection velocity $\boldsymbol{U}_n$. This is understandable from the orbit theory, because a spatial inhomogeneity in $K_\parallel$ and $K_\perp$ produces corresponding inhomogeneities in the magnetic gradient drift (3.24).

The effects of the third term in (5.67) are of special interest. They clearly show that the surfaces of constant density *neither* move with the mean velocity of the guiding centra *nor* with the mass velocity. The reason for this is that compression and expansion effects may occur during the motion. In Chapter 6 we shall make a detailed discussion of the compression effects which arise from div $\boldsymbol{u}_F$ when a force field $\boldsymbol{F}$ drives particles into regions of varying magnetic field strength. The contribution from inertia forces to the third term in equation (5.67) also gives rise to a "piling up" of particles in the form of compression or expansion. When the longitudinal velocity $\boldsymbol{v}_\parallel$ is small, $\boldsymbol{F} = q\boldsymbol{E}$, and $\mathrm{d}\boldsymbol{u}_E/\mathrm{d}t$ can be linearized, we see that this contribution is proportional to $(nm/B^2)\partial(\mathrm{div}\,\boldsymbol{E})/\partial t$. A divergence operation on (2.2) shows that this term can be considered as the rate of change of an electric charge in a medium with an equivalent dielectric constant of the type (3.51). An electric polarization of the plasma is then produced by the inertia forces which act differently on ions and electrons.

We have just seen that surfaces of constant density move with a velocity which may differ both from the guiding centre velocity $\boldsymbol{u}$ and from the mass velocity $\boldsymbol{v}$. According to equations (3.22), (3.43), (3.47), (5.53) and (5.67) it is also clear that there is a difference between the velocities $\boldsymbol{u}$ and $\boldsymbol{v}$. We shall now examine this latter property more closely from the physical point of view.

The mean flux $n\bar{\boldsymbol{u}}_\perp$ of guiding centra in (3.22) differs from the mean flux $n\boldsymbol{v}_\perp$ of particles in (5.53) in that the former contains a magnetic gradient drift, whereas the latter includes a contribution from the pressure gradient. This difference can be explained by the fact that not all particles in a certain volume element have their guiding centra inside the same element, and *vice versa*. To proceed from the flux of guiding centra to the flux of particles

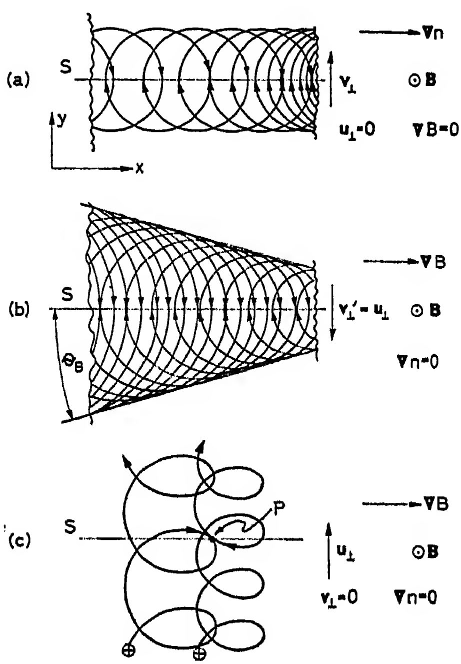


Fig. 5.1. Connexion between the flux of guiding centra and particles through a surface S which is parallel with the magnetic field. Thermal energy assumed to be constant. (a) Homogeneous magnetic field and density gradient directed to the right. (b) Constant density and magnetic field with gradient directed to the right. Particle orbits observed in a frame of reference moving with the guiding centre drift $\boldsymbol{u}_\perp$. (c) The same as (b) but particles are observed in the laboratory frame.

we have to subtract the flux of such centra which have their corresponding particles outside the volume element, and add the flux of such particles which have their guiding centra outside the same element (SPITZER [1952, 1956]). A detailed calculation along these lines leads to the same results as equations (3.47), (3.49) and (5.53).

The reason for a pressure gradient to produce a mass flow is easily understood from Figure 5.1a, where we have assumed the thermal energy $K_\perp$ and the magnetic field to be homogeneous and constant, and that a density gradient exists in the positive x direction. There is no drift velocity of the guiding centra. For a surface S perpendicular to the y axis the figure immediately shows that there are more particles passing upwards through the surface than downwards. Thus, there is a mass motion $\boldsymbol{v}$ perpendicular to $\boldsymbol{B}$ and to ∇n which is generated by the pressure gradient.

In Figures 5.1b and c, however, we instead assume the density n and the thermal energy $K_\perp$ to be constant, whereas there is a constant magnetic gradient ∇B in the positive x direction. This produces a drift $\boldsymbol{u}_\perp = -K_\perp \nabla B \times \boldsymbol{B}/qB^3$ along the y direction. If we choose a frame of reference moving with this velocity the particles will be seen to gyrate around a stationary and homogeneous distribution of guiding centra, as shown in Figure 5.1b. In this frame there is a net flux $n\boldsymbol{v}'_\perp$ of particles through S directed downwards. After some simple geometrical considerations involving the expression (2.81) for the radius of gyration and the magnetic gradient drift (3.24) it is then found that $n\boldsymbol{v}'_\perp$ becomes equal to $-n\boldsymbol{u}_\perp$. These considerations are based on the fact that the magnetic field strength B is inversely proportional to the arc length associated with the angle θ_B in Figure 5.1b, and that the net flux of particles in one direction is equal to one fourth of the mean thermal velocity. Thus, if the observer moves back to the laboratory system, the mass velocity $\boldsymbol{v}_\perp$ will be seen to vanish. This is also easily understood from Figure 5.1c, where the distribution of particles is assumed to be isotropic in phase space. At any point P there are then equal chances to observe particles which have velocities of equal magnitude and opposite directions, and this should be so for all magnitudes and directions. Obviously, the flux of particles does not depend upon the fact whether the paths are curved by a magnetic field or not.

The orbits of Figure 5.1b may also be taken to illustrate a gas of constant density which has a constant gradient $\nabla W \propto \nabla (T_\perp)^{\frac{1}{2}}$ of the transverse "thermal" velocity. The gas is immersed in a homogeneous magnetic field.

In this particular situation a mass velocity $\boldsymbol{v}_\perp$ will be produced by the temperature gradient.

Now consider the extreme case where the magnetic field has a constant modulus B and is curved as shown in Figure 5.2. According to equations

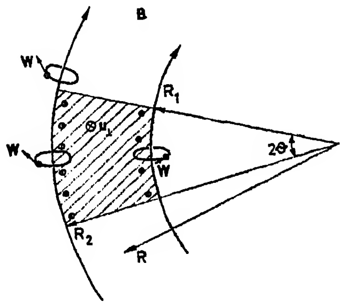


Fig. 5.2. Motion of charged particles in a magnetic field with radius R of curvature.

(3.20) and (3.22) this produces a mean drift into the plane of the figure having the modulus

$$\bar{u}_\perp = 2K_\parallel / eBR, \tag{5.68}$$

where R is the local radius of curvature of the field. We obtain a mean flux of guiding centra through the shaded surface

$$\Psi_u \approx 2n\theta(R_2 - R_1)R\bar{u}_\perp = 4n\theta(R_2 - R_1)K_\parallel / eB \tag{5.69}$$

when $(R_2 - R_1)/R \ll 1$. The flux is defined positive in the direction into the figure. The flux produced by the gyration $\boldsymbol{W}$ at the edges $R = R_1$ and $R = R_2$ of the shaded surface in Figure 5.2 is

$$\Psi_W = -\, n\theta(R_2 - R_1)aW = -\, 2n\theta(R_2 - R_1)K_\perp / eB \tag{5.70}$$

where a is the radius of gyration. In the isotropic case of thermal equilibrium we have $2K_\parallel = K_\perp$ and $\Psi_W = -\Psi_u$ which implies that there is no net flux of particles through the shaded area. This is also expected from the form of the second term of the right hand member in equation (5.53). When there is anisotropy, however, a net flux of particles arises, as is also predicted by the same term.

In the situation of thermal equilibrium we expect from equations (5.20), (5.53) and (3.49) that there should be no mass motions produced by a

gradient in the magnetic field strength. As pointed out by COWLING [1932] and SPITZER [1956] this is also consistent with Liouville's theorem (5.2). Consider namely a mixture of ions and electrons of equal and uniform densities and uniform and isotropic distributions of velocities $\boldsymbol{w}$. The particles are enclosed in a region bounded by perfectly reflecting walls. We assume the initial density of particles in phase space inside the volume to be uniform and the electric space charges to cancel in the initial state. Neglect collisions between particles and assume a stationary magnetic field.

We first discuss only those particles which are uniformly distributed in a region of phase-space with the scalar velocity in the narrow range between w and $w + \Delta w$. According to Liouville's theorem the density of these particles remains constant along a trajectory in phase space. The only force which acts on the particles inside the volume is $q\boldsymbol{w} \times \boldsymbol{B}$ which does not change the modulus w but only the direction $\hat{\boldsymbol{w}}$ of the velocity $\boldsymbol{w}$. The same is true for collisions with the perfectly reflecting walls. Since the initial density is independent of the direction of motion and is also uniform in space, it then follows that the phase-space density will remain so for all later times. Particles in all other ranges Δw of the velocity spectrum will behave in the same way. We therefore conclude that no macroscopic velocities can appear if they do not exist in the initial state, regardless whether there is a magnetic gradient drift or not. The flux of particles reflected from the walls cancels exactly the flux of the guiding centra. This is consistent with conclusions earlier drawn by BOHR [1911] and VAN LEEUWEN [1921].

The present result is understandable also from the point of view of the second law of thermodynamics, which states that the entropy of a closed system cannot decrease. A state of thermal equilibrium of the same system is therefore not expected to develop into a state where mass motions appear. When collisions are taken into account we can consider an initial state with a Maxwellian distribution of velocities. If the gas is in thermodynamic equilibrium with the walls, it is clear from what has just been said that the distribution is self-maintaining and that the macroscopic velocities vanish for all later times.

In conclusion a few comments should be made on the second order contribution to the force $\boldsymbol{F}$ given in (3.40). This contribution makes the external force drift $\boldsymbol{u}_F$ deviate from its zero order value $\boldsymbol{F}_C \times \boldsymbol{B}/qB^2$ given by the force $\boldsymbol{F}_C$ at the guiding centre. One might therefore ask if the Larmor motion produces a change in the mean of the force $\boldsymbol{F}$ "experienced" by ionized matter. The macroscopic equation of motion (5.20) does not indicate

this. Here we shall give a simple example which shows that the contribution $\frac{1}{4}a^2\nabla_\perp^2\boldsymbol{F}$ in (3.40) is exactly cancelled by the difference between the forces acting on the particles and the fictive forces on the guiding centra.

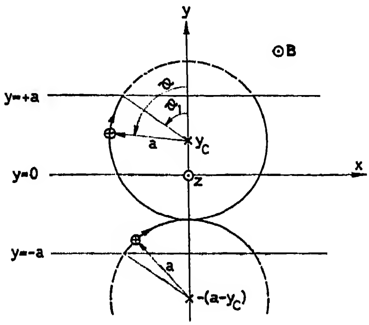


Fig. 5.3. The force on charged particles inside the strip $-a < y < +a$ acts on the particles only during the time intervals where the orbits are situated inside the strip.

Figure 5.3 shows a strip of thickness $2a$ which is infinitely extended in the x and z directions and is immersed in a homogeneous magnetic field in the z direction. The force field $\boldsymbol{F}$ is supposed to have a component and a gradient only along y. With $F = F_y$ a Taylor series expansion around the origin at $y = 0$ yields

$$F = F_0 + \sum_\nu \frac{1}{\nu!}\left[(y_C + a\cos\theta)\frac{\mathrm{d}}{\mathrm{d}y}\right]^\nu F_0, \qquad (5.71)$$

where y_C is the coordinate of the guiding centre at the y axis and subscript ($_0$) indicates the value of F and its derivatives at the origin. If the gyration of a particle around y_C could be neglected the force would become

$$F_C = F_0 + \sum_\nu \frac{1}{\nu!}\left(y_C\frac{\mathrm{d}}{\mathrm{d}y}\right)^\nu F_0 \,. \qquad (5.72)$$

The mean value of this force over the strip is

$$\langle\langle F_C \rangle\rangle = F_0 + \tfrac{1}{6}a^2\frac{\mathrm{d}^2F_0}{\mathrm{d}y^2} \qquad (5.73)$$

in second order.

When the gyration is taken into account it has to be noticed that the force F acts on the particles inside the volume element only during the time which they actually spend there. As a consequence, such parts of the orbits as

the dashed ones in Figure 5.3 will not contribute to the mean force. The mean value of F obtained from integration of (5.71) over a gyro period is

$$\langle F\rangle_+ = \frac{1}{2\pi}\left\{\left[F_0 + y_C\frac{dF_0}{dy} + \left(\tfrac{1}{2}y_C^2 + \tfrac{1}{4}a^2\right)\frac{d^2F_0}{dy^2}\right]\theta\right. \tag{5.74}$$

$$\left. + \left[\frac{dF_0}{dy} + y_C\frac{d^2F_0}{dy^2}\right]a\sin\theta + \tfrac{1}{8}a^2\frac{d^2F_0}{dy^2}\sin 2\theta\right\}_{\theta_1}^{2\pi-\theta_1}$$

for $0 < y_C < 2a$. For $-2a < y_C < 0$ the corresponding expression will be denoted by $\langle F\rangle_-$ and has the limits $-\theta_1$ and θ_1.

Suppose now that the density of particles is constant. The population of guiding centra is then also constant along the y axis and we can calculate the mean values of $\langle F\rangle_+$ and $\langle F\rangle_-$ by integration of $y_C = \pm a - a\cos\theta_1$ from 0 to $2a$ and from $-2a$ to 0, respectively. Thus, the mean force experienced by all particles inside the strip becomes

$$\langle\langle F\rangle\rangle = \frac{1}{2a}\int_0^{2a}\langle F\rangle_+\,dy_C + \frac{1}{2a}\int_{-2a}^{0}\langle F\rangle_-\,dy_C. \tag{5.75}$$

Substitution of (5.74) and the corresponding relation for $\langle F\rangle_-$ into (5.75) yields after some straightforward calculations, that $\langle\langle F\rangle\rangle = \langle\langle F_C\rangle\rangle$ in second order, as expected from the macroscopic theory. This is also reasonable since the instantaneous mean of the force $\boldsymbol{F}$ on the particles cannot depend on their state of motion.

The present conclusions do not contradict the results of Chapter 8, § 2.5 which show that charge separation phenomena are produced by the finite Larmor radius effect in combination with $\nabla_\perp^2\boldsymbol{E}$.

MAGNETIC COMPRESSION

The adiabatic invariants treated in Chapter 4 are closely related to a number of compression and expansion effects which arise from the interactions between ionized matter and a magnetic field. In the present chapter we shall study the corresponding changes in particle density and energy.

1. Connexion between Particle Density and Magnetic Field Strength

The equation of motion of the guiding centre is given by equations (3.16), (3.19), (3.20) and (3.21). When mean values are taken over the velocity distribution it becomes

$$\overline{m\hat{\boldsymbol{u}}_{\|} \frac{\mathrm{d}u_{\|}}{\mathrm{d}t}} + m \overline{\frac{\mathrm{d}\boldsymbol{u}_{\perp}}{\mathrm{d}t}} = q(\boldsymbol{E} + \bar{\boldsymbol{u}} \times \boldsymbol{B}) - m\nabla\phi_{g} - (K_{\perp}/B)\nabla B$$
$$- 2K_{\|}(\nabla_{\perp}B - \hat{\boldsymbol{B}} \times \operatorname{curl}\boldsymbol{B})/B. \quad (6.1)$$

At the same time the mass motion obeys equations (5.20) and (5.24) which can be written as

$$m\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\boldsymbol{v} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) - m\nabla\phi_{g} - \frac{1}{n}\nabla_{\|}(2nK_{\|}) - \frac{1}{n}\nabla_{\perp}(nK_{\perp})$$
$$+ (2K_{\|} - K_{\perp})(\nabla B - \hat{\boldsymbol{B}} \times \operatorname{curl}\boldsymbol{B})/B \quad (6.2)$$

when the pressure tensor is derived from the scalar pressures $p_{\|}$ and $p_{\perp}$.

We shall now examine under what conditions the velocity fields $\boldsymbol{u}$ and $\boldsymbol{v}$ are flux-preserving in the sense of Ch. 2, § 1.3. From the curl of (6.1) follows that

$$\operatorname{curl}(\boldsymbol{E}+\bar{\boldsymbol{u}}\times\boldsymbol{B}) = (1/q)\operatorname{curl}\Big[(K_{\perp}/B)\nabla B+2K_{\|}(\nabla_{\perp}B-\hat{\boldsymbol{B}}\times\operatorname{curl}\boldsymbol{B})/B$$
$$+ \overline{m\hat{\boldsymbol{u}}_{\|} \frac{\mathrm{d}u_{\|}}{\mathrm{d}t}} + m\overline{\frac{\mathrm{d}\boldsymbol{u}_{\perp}}{\mathrm{d}t}}\Big], \quad (6.3)$$

and the curl of (6.2) yields

$$\operatorname{curl}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) = (1/q)\operatorname{curl}\left[\frac{1}{n}\nabla_{\parallel}(2nK_{\parallel}) + \frac{1}{n}\nabla_{\perp}(nK_{\perp}) + (2K_{\parallel} - K_{\perp})(\nabla B - \hat{\boldsymbol{B}} \times \operatorname{curl}\boldsymbol{B})/B + m\frac{\partial \boldsymbol{v}}{\partial t} + m(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right]. \tag{6.4}$$

Generally the conditions (2.29) and (2.34) for flux and line preservation are therefore satisfied neither by the guiding centre drift, nor by the macroscopic fluid velocity. They will only be so in lowest order, i.e. when the right hand members of equations (6.3) and (6.4) can be neglected.

The condition for the drift velocity $\boldsymbol{u}$ of a single particle to become flux preserving may as well be discussed in terms of the curl of equation (3.16) where we put $\boldsymbol{F} = q\boldsymbol{E} - m\nabla\phi_g$. It is satisfied to lowest order where the inertia drift is neglected and it is observed that curl $(M\nabla B)$ vanishes.

The present results indicate that there is in general a certain "slip" between the particles and the field lines. Matter will only become "frozen" to the latter in a first approximation. We now limit the discussion to the lowest order approximation where $\bar{\boldsymbol{u}}$ and $\boldsymbol{v}$ are nearly equal and the conditions leading to (5.55) can be satisfied. Substitution of div $\boldsymbol{v}$ from the equation (5.17) of continuity into (5.57) yields

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{v} + (\boldsymbol{B}/n)\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)n \tag{6.5}$$

or

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)(\boldsymbol{B}/n) = [(\boldsymbol{B}/n)\cdot\nabla]\boldsymbol{v} \tag{6.6}$$

as shown by WALÉN [1946]. Introduce the displacement ξ of a fluid element given by

$$\boldsymbol{v} = \left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\xi, \tag{6.7}$$

and (6.6) can after some deductions be rewritten as

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\{\boldsymbol{B}/n - [(\boldsymbol{B}/n)\cdot\nabla]\xi\} = 0 \tag{6.8}$$

according to LUNDQUIST [1952].

This result can also be deduced from the equation (2.30) for the change of a line element $d\boldsymbol{l}$ which is carried along with the velocity field $\boldsymbol{V}_f$. We

have just shown by means of equations (6.3) and (6.4) that the velocities $\overline{\boldsymbol{u}}$ and $\boldsymbol{v}$ are line- and flux-preserving in zero order where (5.55) is applicable. We can then write

$$\mathrm{d}\boldsymbol{l} = \hat{\boldsymbol{B}}\mathrm{d}l, \quad \mathrm{d}\boldsymbol{l}_0 = \hat{\boldsymbol{B}}_0\mathrm{d}l_0, \quad V_\mathrm{f} = \boldsymbol{v} \approx \overline{\boldsymbol{u}}. \tag{6.9}$$

The line elements are defined positive in the positive direction of $\boldsymbol{B}$ as in Figure 2.3. Now study a volume element of length $\mathrm{d}l_0$ and with the transverse area $\mathrm{d}S_0 = \mathrm{d}\Phi_0/B_0$ which encloses the flux $\mathrm{d}\Phi_0$ as shown in Figure 6.1.

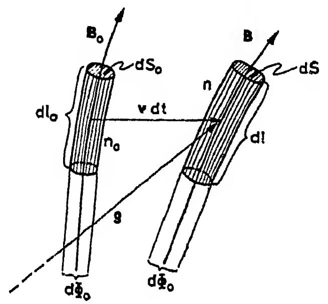


Fig. 6.1. A volume element of particle density n_0 at position $\boldsymbol{\rho} - \boldsymbol{v}\mathrm{d}t$ moves to the position $\boldsymbol{\rho}$ where the density becomes n (cf. LUNDQUIST [1952]).

This element contains a certain number of particles. During the motion the particles will remain inside an element which encloses a constant magnetic flux. After a time $\mathrm{d}t$ they have moved the distance $\boldsymbol{v}\mathrm{d}t$ and form a new element of length $\mathrm{d}l$ and area $\mathrm{d}S = \mathrm{d}\Phi_0/B$. Since the particle number is constant we have

$$n\,\mathrm{d}l/B = n_0\mathrm{d}l_0/B = \text{const.} \tag{6.10}$$

Combination of equations (2.30), (6.9) and (6.10) then gives

$$\boldsymbol{B}/n = \boldsymbol{B}_0/n_0 + [(\boldsymbol{B}_0/n_0)\cdot\nabla]\boldsymbol{v}\,\mathrm{d}t. \tag{6.11}$$

Equation (6.7) implies that the displacement $\xi(\rho, t)$ is defined in such a way that $\rho - \xi$ is the position at time $t = 0$ of a particle which is at ρ at time t. Further if we choose $\xi = 0$ when the element is at ρ and $\xi(t = 0) = = \xi_0 = -\boldsymbol{v}\,\mathrm{d}t$, the vector $\boldsymbol{v}\,\mathrm{d}t$ will have the direction given in Figure 6.1. It is therefore easily seen that equation (6.11) is equivalent to the expression which results from integration of (6.8) (cf. LUNDQUIST [1952]).

For a motion which is flux- and line-preserving we have just found that the magnetic field and the particle density are connected by relations such as

equations (6.8), (6.10) and (6.11). There is still another way in which this becomes obvious. In zero order (5.67) reduces to (5.63), provided that we can justify the assumptions about the pressure tensor which were specified in connexion with equation (5.55). Observe that $(1/q)$ curl $\boldsymbol{F} = -\partial \boldsymbol{B}/\partial t$ according to equations (2.37) and (2.1). After some vector operations we can write (5.63) in the form

$$\frac{\partial n}{\partial t} - \frac{n}{B} \cdot \frac{\partial B}{\partial t} - (1/q)\boldsymbol{F} \cdot [(n/B^2)\,\mathrm{curl}\,\boldsymbol{B} - \boldsymbol{B} \times \nabla(n/B^2)]$$
$$+ \boldsymbol{v}_{\parallel} \cdot \nabla n + nB(\hat{\boldsymbol{v}}_{\parallel} \cdot \nabla)(v_{\parallel}/B) = 0, \qquad (6.12)$$

where the expression (3.23) for $\boldsymbol{u}_F$ has been inserted. The terms which include $\boldsymbol{v}_{\parallel}$ generally produce longitudinal convection and compression effects. They disappear when the flow along $\boldsymbol{B}$ is incompressible or when ∇n has no component along $\boldsymbol{B}$. Restrict the discussion to the case $\boldsymbol{v}_{\parallel} = 0$ and study the following situations in a stationary magnetic field:

(i) Suppose that the electric currents can be neglected in the region where the particles move. Then curl $\boldsymbol{B} = 0$ and equation (6.12) becomes

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}_F \cdot \nabla\right)(n/B^2) = 0, \quad \boldsymbol{v}_{\parallel} = 0. \qquad (6.13)$$

This implies that n/B^2 is constant during the motion which takes place across $\boldsymbol{B}$ at the velocity $\boldsymbol{u}_F$. That n is proportional to the square of B is explained by the fact that both the cross section dS and the length dl of a volume ele-

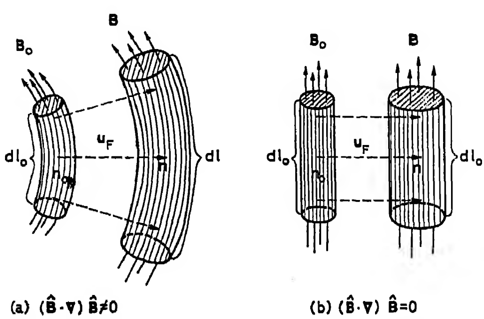


Fig. 6.2. Expansion of ionized matter moving across an inhomogeneous magnetic field. No longitudinal motion is taking place ($\boldsymbol{v}_{\parallel} = 0$). (a) Curved field lines. (b) Straight field lines.

ment increase inversely proportional to B when the particles move from a stronger field $\boldsymbol{B}_0$ to a weaker field $\boldsymbol{B}$ as demonstrated in Figure 6.2a. That $\mathrm{d}l$ is proportional to $1/B$ depends upon the curvature of the field lines; this is also in agreement with equation (6.10).

(ii) If the field lines are straight (6.12) instead reduces to

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}_F \cdot \nabla\right)(n/B) = 0, \quad \boldsymbol{v}_{\|} = 0 \tag{6.14}$$

which shows that n/B will now become a constant of the motion. The reason for this is that the length $\mathrm{d}l$ does not vary when the field lines are straight as shown in Figure 6.2b. Also this agrees with equation (6.10). In a coordinate system moving with the velocity $\boldsymbol{u}_f$ the density of field lines per unit area of the cross section $\mathrm{d}S$ is therefore seen to vary at the same rate as the density of particles.

2. Compression and Heating Mechanisms

A number of compression and heating mechanisms are intimately connected with the constancy of the adiabatic invariants M and J. The former is related to a transverse compression and the latter to a longitudinal one. Such compression (or expansion) processes occur in an ionized gas which is trapped in a time-dependent magnetic field. The field lines then act like pistons which push the gas. It is not necessary, however, that the field varies in time to produce density changes in the gas. This has already been demonstrated by the examples of § 1 and Figure 6.2 where the gas was forced by a drift motion into regions of variable magnetic field strength. We shall here investigate some of the compression mechanisms more in detail.

2.1. MAGNETIC MIRROR COMPRESSION

Assume as in Chapter 4 that, in absence of electrostatic and gravitation fields, a particle is trapped in the magnetic field of Figure 6.3, i.e. between two mirror points $s = s_1$ and $s = s_2$ on a field line. The strength $B = B(s, t)$ is a function of the longitudinal coordinate s. When the field $\boldsymbol{B}$ varies slowly compared to the gyro time t_g and the time $t_{\|}$ of longitudinal oscillations between the mirrors the equivalent magnetic moment

$$M = mW^2/2B(s, t) \tag{6.15}$$

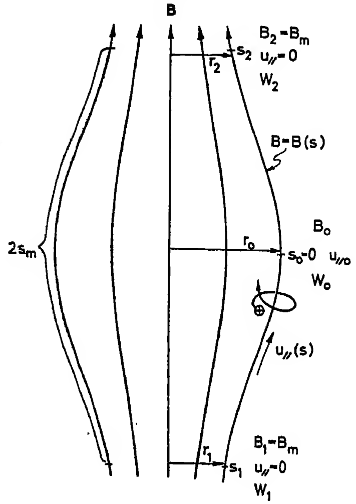


Fig. 6.3. Particle moving along magnetic field line between two mirror points at s_1 and s. The cross section with the weakest field B_0 is given by r_0.

and the longitudinal invariant

$$J = m \oint u_{\|} \,\mathrm{d}s = 2m \int_{s_1}^{s_2} u_{\|} \,\mathrm{d}s \tag{6.16}$$

become approximate constants of the motion.

With these starting points we shall now discuss the compression phenomena which occur in the field $\boldsymbol{B}$, essentially on the basis of results earlier derived by POST [1958]. First consider the magnetic moment M. Since the characteristic time $A/|\partial \boldsymbol{A}/\partial t|$ of the magnetic field is long compared to t_g and $t_{\|}$ the right hand member of (2.38) should become small. Thus, the total velocity w is given by

$$w^2 \approx u_{\|}^2 + W^2 \approx W_m^2. \tag{6.17}$$

This quantity is approximately constant during a period $t_{\|}$, but not during times comparable to the magnetic field changes. In (6.17) $u_{\perp} \ll W$ and W_m is the velocity of gyration at the turning points s_1 and s_2, where $u_{\|} = 0$. According to (6.15) the corresponding field strengths B_1 and $B_2 = B_m$ at two successive reflections will then become equal. They are related to W by

$$W^2/B(s, t) = W_0^2/B_0(t) = W_m^2/B_m(t) = 2M/m = \text{const.} \tag{6.18}$$

Here subscript ($_0$) refers to the "equatorial" plane where the field $\boldsymbol{B}$ has its lowest value. Combination of equations (6.17) and (6.18) yields

$$u_{\|}^2 = W_{\mathrm{m}}^2(1 - B/B_{\mathrm{m}}). \tag{6.19}$$

Since $u_{\|}^2$ is positive the particle cannot reach regions of fields exceeding $B = B_{\mathrm{m}}$. Equation (6.19) is the well-known relation for magnetic mirror reflection.

Introduce the *mirror ratio*

$$R(t) = B(t)/B_0(t), \quad R_{\mathrm{m}} = B_{\mathrm{m}}/B_0 \tag{6.20}$$

and equations (6.17) and (6.18) combine to

$$w^2(t) = w^2(0) \cdot \frac{R_{\mathrm{m}}(t)}{R_{\mathrm{m}}(0)} \cdot \frac{B_0(t)}{B_0(0)} \tag{6.21}$$

which gives the increase in energy in terms of the longitudinal and transverse compression parameters.

Before proceeding to the main compression problem we shall draw attention to a certain detail in the structure of the particle orbit. The magnetic flux encircled by the Larmor motion during a gyro period is $\pi a^2 B = 2(m/q^2)M$ which is approximately constant as pointed out in Ch. 4, § 1.2. On the other hand the particle touches two field lines in the "equatorial" plane at the points r_0 and $r_0 + 2a_0$. A corresponding annular region encloses the flux $\Delta\Phi_0 \approx 4\pi r_0 a_0 B_0$ as shown in Figure 6.4. When the particle has moved

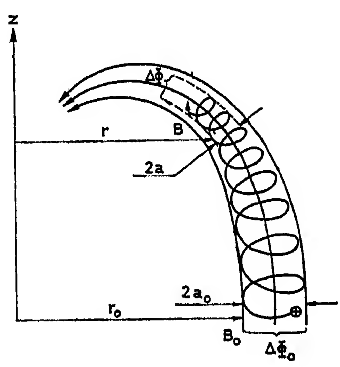


Fig. 6.4. A particle moving in a mirror field does not touch the same field lines all the time. Figure shows example when B increases more rapidly than $1/r^2$.

along $\boldsymbol{B}$ to a new position it touches instead the dashed field lines in the figure, and the corresponding flux is $\Delta\Phi \approx 4\pi raB$. Thus, $\Delta\Phi/\Delta\Phi_0 = (r/r_0) \cdot \cdot (B/B_0)^{\frac{1}{2}}$. This implies that the particle touches the same field lines during its gyration only when $B \propto 1/r^2$. This happens for a monopole field like that in Figure 2.6, but not for a mirror field of general shape. Thus, it does not occur when the particle gyrates around a field line which is off the axis of symmetry, as is the case in Figure 6.4.

We further turn to the longitudinal invariant J. From equations (6.16) – (6.18) and (6.20) we have

$$J = 2m \int_{s_1}^{s_2} (W_m^2 - W^2)^{\frac{1}{2}} ds = (8mM)^{\frac{1}{2}} \int_{s_1}^{s_2} (B_m - B)^{\frac{1}{2}} ds$$

$$= [8mMB_0(t)]^{\frac{1}{2}} \int_{R_m(1)}^{R_m(2)} [R_m(t) - R(s, t)]^{\frac{1}{2}} \frac{ds}{dR} dR \tag{6.22}$$

when B changes slowly during a longitudinal period $t_{\|}$. From this result we can determine how the mirror points s_1 and s_2 move during the compression. In combination with (6.21) it gives complete information about the changes in density and energy. This is illustrated by three examples:

(i) Suppose that the shape of the magnetic field is unchanged during the compression and is determined by

$$R(s, t) = B(s, t)/B_0(t) = 1 + (s/s_B)^2, \tag{6.23}$$

where s_B is a constant. Equation (6.22) then becomes

$$J = \pi s_m^2(t) \cdot [2mMB_0(t)]^{\frac{1}{2}}/s_B, \tag{6.24}$$

where $2s_m(t)$ is the distance along a field line between the mirror points at time t. The constancy of J then gives a longitudinal compression factor

$$\varkappa_{\|} = \frac{s_m(0)}{s_m(t)} = \left[\frac{B_0(t)}{B_0(0)}\right]^{\frac{1}{4}} \tag{6.25}$$

and a relative change in the mirror ratio determined by

$$\frac{[R_m(t) - 1]^2}{[R_m(0) - 1]^2} = \frac{B_0(0)}{B_0(t)}. \tag{6.26}$$

Especially if we assume a small mirror ratio the strength B will become approximately proportional to $1/r^2$. This holds along a given field line and

during the compression process which is nearly flux-preserving according to § 1. We can then write

$$\varkappa_\perp \approx B_0(t)/B_0(0) \tag{6.27}$$

for the corresponding compression factor in the transverse direction and

$$\varkappa = \varkappa_\parallel \cdot \varkappa_\perp = n(t)/n(0) = \left[B_0(t)/B_0(0)\right]^{5/4} \tag{6.28}$$

for the total compression factor. This determines the relative increase $n(t)/n(0)$ in density. The increase in energy is given by equations (6.21) and (6.26):

$$\frac{w^2(t)}{w^2(0)} = \frac{B_0(t)}{B_0(0)} \cdot \left\{\left[\frac{B_0(0)}{B_0(t)}\right]^{\frac{1}{2}} \cdot \left[1 - \frac{B_0(0)}{B_m(0)}\right] + \frac{B_0(0)}{B_m(0)}\right\}. \tag{6.29}$$

From equation (6.25) we see that the distance $2s_m$ between the mirror points shrinks, even if the shape of the magnetic field is preserved during the compression. This is understandable since there are no collisions which directly couple the longitudinal and transverse motions. The latter is subject to a more powerful, two-dimensional compression than the former, and W^2 therefore increases more rapidly than $u_\parallel^2$. This also agrees with the result that $\varkappa_\perp > \varkappa_\parallel$. As a consequence, the repulsive force by W^2 in the longitudinal direction (cf. Fig. 3.1) increases more rapidly than the "pressure" by $u_\parallel^2$ and the mirror points move closer together.

(ii) If there are no radial changes in the magnetic field and only the distance between the magnetic mirrors is changed a one-dimensional compression occurs like that suggested by FERMI [1954]. A very simple picture of this has been given in Ch. 4, § 1.3 in connexion with Figure 4.4. In such a process $\varkappa_\perp$ is constant and $\varkappa_\parallel$ increases inversely proportional to $s_m(t)$. Equation (4.21) then shows that the increase in energy is given by the acceleration of the longitudinal velocity, i.e. $W(t) = W(0)$ and

$$\frac{w^2(t) - W^2(0)}{w^2(0) - W^2(0)} = \frac{u_\parallel^2(t)}{u_\parallel^2(0)} = \frac{s_m^2(0)}{s_m^2(t)}. \tag{6.30}$$

(iii) A magnetic field which changes its shape in a general way during the compression can in principle be treated by the present equations, but detailed calculations become rather involved. Here we restrict ourselves to the simple case where R is a function g_B such that

$$R = \frac{B(s, t)}{B_0(t)} \equiv g_B = g_B\left[\frac{s}{s_B(t)}\right] \tag{6.31}$$

and $s_B(t)$ changes during the compression in such a way that R is independent of t for all points between s_1 and s_2. The behaviour of g_B is then representable by a simple scaling of the longitudinal coordinate of the field. This is what would happen if the mirror ratios are left fixed, at the same time as the mirrors are moved together slowly. With g_B^{-1} as the inverse function of $g_B(s/s_B)$ we have $\mathrm{d}s/\mathrm{d}R = s_B \mathrm{d}(g_B^{-1})/\mathrm{d}R$ and (6.22) becomes

$$J = s_B(t) \cdot [8mMB_0(t)]^{\frac{1}{2}} \int_{R_m(1)}^{R_m(2)} [R_m - R(s)]^{\frac{1}{2}} \frac{\mathrm{d}(g_B{}^{-1})}{\mathrm{d}R} \mathrm{d}R. \tag{6.32}$$

But g_B^{-1} and the square root of the integrand are functions only of the mirror ratios $R(s)$ which are assumed to be independent of time for all points s inside the configuration. Therefore the integrand is no longer an explicit function of time.

Of special interest is a situation where the values of R_m remain constant for all trapped particles of an ionized gas. The constancy of J then also requires $s_B{}^2 B_0$ to be constant for the same particles. Since the cross sectional area of a flux tube is proportional to $1/B$ this implies that the compression is uniform in all directions. Assume the mirror ratio to be small so that (6.27) holds. When R_m is independent of time the energy increases proportionally to the transverse compression factor,

$$\frac{w^2(t)}{w^2(0)} = \frac{B_0(t)}{B_0(0)} \approx \varkappa_\perp, \tag{6.33}$$

as seen from equations (6.21) and (6.27). However, the volume of the gas varies as $(\varkappa_\parallel \cdot \varkappa_\perp)^{-1}$ and therefore as $(\varkappa_\perp)^{-\frac{3}{2}}$, since

$$\varkappa_\parallel = \varkappa_\perp^{\frac{1}{2}} = \left[\frac{B_0(t)}{B_0(0)}\right]^{\frac{1}{2}}. \tag{6.34}$$

Consequently, the mean particle energy varies inversely with the $\frac{2}{3}$ power of the volume. As expected, this agrees with the adiabatic law (5.60) for isotropic compression of a gas with the ratio $\frac{5}{3}$ between the specific heats.

2.2. ALFVÉN'S HEATING MECHANISM

In the preceding paragraph we have seen how the energy of the particles

can be increased by magnetic compression in a field which varies in time and "squeezes" the gas in the transverse and longitudinal directions. An alternative method has been suggested by ALFVÉN [1954a] by which compression of the gas is achieved in a stationary magnetic field. This can be done if the particles are forced to move into regions of increasing magnetic field strength.

To simplify the discussion we consider the somewhat artificial case of an inhomogeneous magnetic field with straight field lines, like that given in Figure 6.2b. From equations (3.22) and (3.24) of the orbit theory we see that the magnetic gradient drift cannot force the particles into a stronger magnetic field because it is directed along surfaces B = const. However, the same equations, as well as (6.14), also show that a motion into an increasing field B will become possible if a force field $\boldsymbol{F}$ is applied in such a way that the corresponding drift velocity $\boldsymbol{u}_F$ has a component directed across the surfaces B = const. In the present two-dimensional case (6.14) and constancy of the magnetic moment M require that

$$n/n_0 = B/B_0 = W^2/W_0^2. \tag{6.35}$$

This applies to an element of the gas which starts at a point where the field strength is B_0 and where the density is n_0 and the energy of gyration $K_{\perp 0} = \frac{1}{2}mW_0^2$. The compression which occurs is two-dimensional and (6.35) shows that $K_\perp$ is proportional to n, as expected for adiabatic changes of a system with two degrees of freedom.

To illustrate the present mechanism in detail, consider the example of

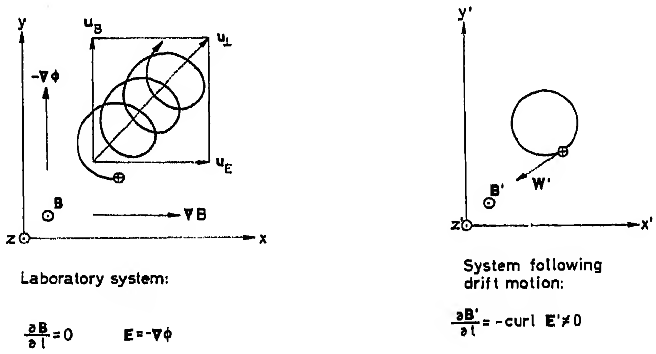


Fig. 6.5. Particle moving in inhomogeneous magnetic field $\boldsymbol{B}$ and crossed electric field $\boldsymbol{E}$. Left hand part of figure gives situation as seen from the laboratory system and right hand part the situation seen by an observer moving with the drift velocity $\boldsymbol{u}_\perp$.

Figure 6.5. Here a particle moves in the crossed fields $\boldsymbol{B}$ and $\boldsymbol{E}$ given by

$$\boldsymbol{B} = (0, 0, B_0(1 + x/x_0)), \quad \boldsymbol{E} = (0, E_0, 0) \tag{6.36}$$

in rectangular coordinates; B_0, x_0 and E_0 are constants and $x_0 \gg a$. The total drift velocity has a constant component $\boldsymbol{u}_\parallel$ in the longitudinal direction and a transverse component

$$\boldsymbol{u}_\perp = (E_0/B_0,\ M/qx_0,\ 0), \tag{6.37}$$

which also becomes constant in this simple case. The electric field drift E_0/B_0 is in the x direction, along ∇B, and the particle can now move into a magnetic field of increasing intensity by which a compression occurs. The situation is also understood from the fact that the magnetic gradient drift, M/qx_0, moves the particle in the y direction along $\boldsymbol{E}$. Thus the particle "falls" in the electric field and gains energy.

If we observe the motion from the laboratory system, we see how the particle gains energy by drifting across the electric equipotential surfaces. Since $\boldsymbol{u}_\perp$ is constant the mean increase in energy over a Larmor period becomes

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}mw^2) \right\rangle = \left\langle \frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}mW^2) \right\rangle = \langle q\boldsymbol{E} \cdot (\boldsymbol{u} + \boldsymbol{W}) \rangle$$

$$= q\boldsymbol{E} \cdot \boldsymbol{u} = ME_0/x_0 \tag{6.38}$$

according to equation (2.38).

If we instead place ourselves in a coordinate system moving with the velocity $\boldsymbol{u}_\perp$, we will see a particle which only performs a Larmor motion $\boldsymbol{W}'$ in a magnetic field $\boldsymbol{B}'$ which changes in time. The electric field $\boldsymbol{E}'$ measured in this system obeys the equation

$$\operatorname{curl} \boldsymbol{E}' = -\frac{\partial \boldsymbol{B}'}{\partial t}. \tag{6.39}$$

If relativistic effects are neglected the observed changes of the field $\boldsymbol{B}'$ are related to $\boldsymbol{B}$ and the rate $\boldsymbol{u}_\perp$ of displacement by

$$\frac{\partial \boldsymbol{B}'}{\partial t} = (\boldsymbol{u}_\perp \cdot \nabla)\boldsymbol{B} = \left(0, 0, u_x \frac{\mathrm{d}}{\mathrm{d}x}\left(B_0 \frac{x}{x_0}\right)\right) = \left(0, 0, \frac{E_0}{x_0}\right). \tag{6.40}$$

As seen from the moving coordinate system the mean increase in energy by magnetic compression becomes

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}mW'^2\right)\right\rangle = \left\langle q\boldsymbol{E}'\cdot\boldsymbol{w}'\right\rangle = \frac{q}{t_g}\int_0^{t_g}\boldsymbol{E}'\cdot\boldsymbol{w}'\mathrm{d}t$$
$$= -\frac{q}{t_g}\oint \boldsymbol{E}'\cdot\mathrm{d}\boldsymbol{l} = -\frac{q}{t_g}\iint(\mathrm{curl}\,\boldsymbol{E}')\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S \qquad (6.41)$$
$$= \frac{q}{t_g}\iint\frac{\partial\boldsymbol{B}'}{\partial t}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S \approx \frac{ME_0}{x_0}$$

during a Larmor period $t_g = 2\pi/\omega_g$. Integration is performed around the Larmor orbit with the radius a and area πa^2. In (6.41) we observe that $\boldsymbol{w}' = -\,\mathrm{d}\boldsymbol{l}/\mathrm{d}t$ when $q > 0$ and the normal $\hat{\boldsymbol{n}}$ is defined in the conventional way. The result (6.41) agrees with (6.38) which was deduced from the motion in the laboratory system.

The compression work per unit volume obtained from (6.41) should be equal to the work against the pressure $p_\perp = \frac{1}{2}nmW^2$ when the specific volume $1/n$ is being changed:

$$\frac{ME_0}{x_0} = -\tfrac{1}{2}nmW^2\,\frac{\partial}{\partial t}\left(\frac{1}{n}\right). \qquad (6.42)$$

Combination with (6.40) and the expression $M = mW^2/2B$ then shows that n/B should be constant during the compression, as expected.

*2.3. THE GYRORELAXATION EFFECT

So far we have not discussed the influence of collisions on the present compression mechanisms. We shall here use a simplified model where Coulomb collisions are treated as discrete binary events. Between the collisions the adiabatic invariance can then still be assumed to hold. The collisions provide an additional coupling between the longitudinal and transverse velocities which tends to establish an isotropic velocity distribution. ALFVÉN [1954a] has suggested that particles may pass repeatedly through regions with different strengths of the magnetic field. The mechanism of the preceding paragraph, which is reversible in absence of collisions, will then produce a net gain of heat during each cycle when collisions are operative. We shall study this situation a little more in detail and follow the lines of an earlier paper by SCHLÜTER [1957].

Start with an initial state of thermal equilibrium, where the velocity distribution is isotropic so that $p_{\parallel 0} = 2nK_{\parallel 0} = nmu_{\parallel 0}^2$ is equal to $p_{\perp 0} = nK_{\perp 0} = \frac{1}{2}nmW_0^2$. Suppose that there is a magnetic field which changes in time and

produces compression effects which disturb the isotropy. Collisions will on the other hand tend to reestablish isotropy, and this effect will increase the larger the deviation $W^2 - 2u_\parallel^2$ becomes. Assume the rate of change of this quantity towards its zero equilibrium value to be given simply by $-\gamma_c(W^2 - 2u_\parallel^2)$ where γ_c is a positive constant.

Between the collisions the magnetic moment $M = mW^2/2B$ is constant and in absence of collisions W^2 would increase at a rate given by $(W^2/B)\cdot(\mathrm{d}B/\mathrm{d}t)$. We restrict ourselves to straight field lines where there is no longitudinal compression. Energy is then fed into the system from $\mathrm{d}B/\mathrm{d}t$ only by a transverse compression which changes W.

Now assume the magnetic compression and the collisions to be present simultaneously and superimpose their effects. For the change in the total energy $\frac{1}{2}mw^2 \approx \frac{1}{2}m(W^2 + u_\parallel^2)$ follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(W^2 + u_\parallel^2) = \frac{W^2}{B}\cdot\frac{\mathrm{d}B}{\mathrm{d}t} \tag{6.43}$$

since the magnetic field is the only external source which feeds energy into the system as a whole. The collisions will cause a "flow" of this energy inside the system between the longitudinal motion $u_\parallel$ and the Larmor motion W. On the other hand, the quantity $W^2 - 2u_\parallel^2$ should decrease by means of collisions at the same time as energy is fed into the Larmor motion W by $\mathrm{d}B/\mathrm{d}t$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(W^2 - 2u_\parallel^2) = -\gamma_c(W^2 - 2u_\parallel^2) + \frac{W^2}{B}\cdot\frac{\mathrm{d}B}{\mathrm{d}t}. \tag{6.44}$$

Combination of equations (6.43) and (6.44) yields

$$\frac{\mathrm{d}W^2}{\mathrm{d}t} = -\tfrac{1}{3}\gamma_c(W^2 - 2u_\parallel^2) + \frac{W^2}{B}\cdot\frac{\mathrm{d}B}{\mathrm{d}t}, \tag{6.45}$$

$$\frac{\mathrm{d}u_\parallel^2}{\mathrm{d}t} = \tfrac{1}{3}\gamma_c(W^2 - 2u_\parallel^2). \tag{6.46}$$

Equation (6.46) does not contain $\mathrm{d}B/\mathrm{d}t$. This is expected because the longitudinal motion receives all its energy from the diffusion in velocity space by collisions.

We consider the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{W^4u_\parallel^2}{B^2}\right) = \frac{W^2}{B^2}\left(2u_\parallel^2\,\frac{\mathrm{d}W^2}{\mathrm{d}t} + W^2\,\frac{\mathrm{d}u_\parallel^2}{\mathrm{d}t} - 2\,\frac{W^2u_\parallel^2}{B}\,\frac{\mathrm{d}B}{\mathrm{d}t}\right). \tag{6.47}$$

In combination with equations (6.45) and (6.46) it yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{W^4 u_{\parallel}^2}{B^2}\right) = \gamma_c W^2 \frac{(W^2 - 2u_{\parallel}^2)^2}{3B^2} > 0 \tag{6.48}$$

since γ_c is a positive quantity.

The result shows that when the magnetic field oscillates and after one cycle returns to its initial value the quantity $W^4 u_{\parallel}^2$ will increase during the same cycle, as well as the energy. Thus, an oscillating magnetic field provides a heating mechanism. The situation resembles that of an elastic body which has internal friction and is subject to pulsating deformations. The body will exhibit hysterisis in such a way that part of the applied work during a cycle goes into heat. The result of (6.48) arises from analogous conditions and is expected from the second law of thermodynamics.

* 3. Magnetohydrodynamic Waves

The electrodynamic force which is generated by induced currents in an electrically conducting fluid often acts like the restoring force of an elastic string. It is therefore imaginable that this force may interact with the inertia of the fluid in a way to produce wave phenomena. Such magnetohydrodynamic waves were discovered by ALFVÉN [1942]. An additional restoring force is due to the pressure gradient which in absence of magnetic fields determines the mechanism of ordinary sound waves. In presence of a magnetic field a mixture of wave types arises where both electrodynamical and mechanical effects contribute to the restoring force. Such waves were first discussed by HERLOFSON [1950] and HOFFMANN and TELLER [1950]. Further considerations of this subject are due to ÅSTRÖM [1950, 1956] and VAN DE HULST [1951] among others.

Here we shall study a plane magnetohydrodynamic wave propagating in a homogeneous plasma and in a homogeneous and static external field $\boldsymbol{B}_0$. The undisturbed state of the gas is given by a constant density n_0 of ions and electrons which have the constant thermal energies $K_{i\parallel} = K_{e\parallel} = K_{\parallel}$ and $K_{i\perp} = K_{e\perp} = K_{\perp}$ in the longitudinal and transverse directions. The force field is due to an electric field $\boldsymbol{E}$ which is generated by the wave itself. The wave motion also induces an electric current density $\boldsymbol{j}$ and a magnetic field $\tilde{\boldsymbol{B}} = \boldsymbol{B} - \boldsymbol{B}_0$, where $\boldsymbol{B}$ is the total field strength. Restrict the treatment to small wave amplitudes where $\tilde{B} \ll B_0$. We further consider a plane wave and assume an arbitrary direction of its wave number $\boldsymbol{\kappa}$ with respect to the field $\boldsymbol{B}_0$. The study will include such modes which are associated both with a

compression and a transverse deformation of the magnetic field.

Combination of (5.20) and (2.2) yields

$$nm_i \frac{\partial \boldsymbol{v}}{\partial t} = \frac{1}{\mu_0} \operatorname{curl} \boldsymbol{B} \times \boldsymbol{B} - \operatorname{div} \pi, \tag{6.49}$$

where the inertia of electrons and the displacement current have been neglected, $n_i \approx n_e \equiv n$, $\boldsymbol{v}$ is the velocity of the centre of mass and π denotes the sum of the ion and electron pressure tensors. We take the time derivative of (6.49) and substitute (5.57) into the resulting expression:

$$nm_i \frac{\partial^2 \boldsymbol{v}}{\partial t^2} = \frac{1}{\mu_0} [\operatorname{curl} \operatorname{curl} (\boldsymbol{v} \times \boldsymbol{B})] \times \boldsymbol{B} - \frac{\partial}{\partial t} (\operatorname{div} \pi). \tag{6.50}$$

The lowest order form (5.56) of Ohm's law has been used in this deduction. The approximation holds in the present situation, which is controlled by magnetic induction effects and not by charge separation phenomena.

A coordinate system is now chosen where the wave normal is in the yz plane and where all quantities vary as $\exp [i(\kappa_y y + \kappa_z z + \omega t)]$. Consider adiabatic changes of state:

(i) In the isotropic case the pressure tensor is represented by a scalar pressure $p = p_i + p_e$. The unperturbed value P of the latter corresponds to a sound velocity U_s given by $U_s^2 = 5P/3nm_i$. From (5.60) and (5.17) we obtain

$$\frac{\partial p}{\partial t} = -\frac{5}{3} P \operatorname{div} \boldsymbol{v}. \tag{6.51}$$

After substitution of (6.51) into (6.50) the corresponding dispersion relations yield three types of waves. The first is associated with $\tilde{B}_x$ and v_x and has a phase velocity U_p given by

$$U_p^2 = \omega^2/(\kappa_y^2 + \kappa_z^2) = V_A^2 \cos^2\theta, \quad V_A^2 = B_0^2/\mu_0 nm_i, \tag{6.52}$$

where V_A is the Alfvén velocity and $\cos^2\theta = \kappa_z^2/(\kappa_y^2 + \kappa_z^2)$. The second and third waves are associated with $\tilde{B}_y$, $\tilde{B}_z$ and v_y, v_z and for their phase velocities we have (cf. THOMPSON [1962])

$$U_p^2 = \tfrac{1}{2} (V_A^2 + U_s^2) \pm \tfrac{1}{2} [(V_A^2 + U_s^2)^2 - 4 V_A^2 U_s^2 \cos^2\theta]^{\frac{1}{2}}. \tag{6.53}$$

(ii) For anisotropic changes of state we follow some earlier deductions by LÜST [1960] and use the double adiabatic theory of chapter 5, § 1.3 and

§ 2.2. Introduce the sound velocities $U_{s\parallel}$ and $U_{s\perp}$ given by $U_{s\parallel}^2 = 3P_{\parallel}/nm_i$ and $U_{s\perp}^2 = 2P_{\perp}/nm_i$ where $P_{\parallel}$ and $P_{\perp}$ are the unperturbed values of the longitudinal and transverse pressures $p_{\parallel}$ and $p_{\perp}$. Expressions for $\partial p_{\parallel}/\partial t$ and $\partial p_{\perp}/\partial t$ are obtained from equations (5.34) and (5.35) and are substituted into expression (5.24) for the pressure tensor. The derivatives of $\boldsymbol{B}$ included in the latter are further written in terms of $\boldsymbol{v}$ by means of (3.21) and (5.57). The result becomes

$$\frac{\partial}{\partial t}(\operatorname{div}\pi)_x = -(P_{\parallel} - P_{\perp})\,\kappa_z^2 v_x, \tag{6.54}$$

$$\frac{\partial}{\partial t}(\operatorname{div}\pi)_y = P_{\perp}\kappa_y(2\kappa_y v_y + \kappa_z v_z) - (P_{\parallel} - P_{\perp})\,\kappa_z^2 v_y, \tag{6.55}$$

$$\frac{1}{\partial t}(\operatorname{div}\pi)_z = P_{\parallel}\kappa_z(\kappa_y v_y + 3\kappa_z v_z) - (P_{\parallel} - P_{\perp})\,\kappa_y\kappa_z v_y. \tag{6.56}$$

With (6.54) – (6.56) inserted into (6.50) we obtain one wave associated with $\tilde{B}_x$ and v_x where

$$U_p^2 = \left(V_A^2 - \tfrac{1}{3}U_{s\parallel}^2 + \tfrac{1}{2}U_{s\perp}^2\right)\cos^2\theta \tag{6.57}$$

and two waves associated with $\tilde{B}_y$, $\tilde{B}_z$ and v_y, v_z where

$$\begin{aligned} U_p^2 = \tfrac{1}{2}\left[V_A^2 + \tfrac{2}{3}U_{s\parallel}^2\cos^2\theta + U_{s\perp}^2\left(1 - \tfrac{1}{2}\cos^2\theta\right)\right] \pm \\ \pm \tfrac{1}{2}\left\{\left[V_A - \tfrac{4}{3}U_{s\parallel}^2\cos^2\theta + U_{s\perp}^2\left(1 - \tfrac{1}{2}\cos^2\theta\right)\right]^2 + U_{s\perp}^4\cos^2\theta\sin^2\theta\right\}^{\frac{1}{2}}. \end{aligned} \tag{6.58}$$

When $\theta = \pi/2$ a longitudinal wave results in both of cases (i) and (ii). It is a compression wave shown in Figure 6.6b where the "elastic" forces due to the electromagnetic field and the pressure gradient are superimposed. In (6.53) the total pressure P contributes to this force, whereas only the transverse pressure $P_{\perp}$ gives the corresponding contribution in (6.58).

If instead $\theta = 0$ equation (6.52) and the first solution (6.53) yield a transverse Alfvén wave where only the electromagnetic field produces a restoring force as shown in Figure 6.6c. The second solutions (6.53) and (6.58) are pure sound waves where a three-dimensional compression takes place in case (i) and a one-dimensional in case (ii). Further, equation (6.57) and the first solution (6.58) become equal and represent an Alfvén wave which is modified by the anisotropy in pressure. If

$$P_{\parallel} > B^2/\mu_0 + P_{\perp} \tag{6.59}$$

an Alfvén wave (or "fire-hose") instability occurs, as first found by PARKER [1958]. It is due to the centrifugal force from the thermal motion along the wave-shaped magnetic field lines.

For an arbitrary direction of θ the second solution (6.58) becomes negative when

$$P_{\perp}^2 \sin^2\theta > 3P_{\parallel}[(B^2/\mu_0) - P_{\parallel}\cos^2\theta + P_{\perp}(1 + \sin^2\theta)]. \tag{6.60}$$

This gives rise to a mirror instability which has earlier been treated more rigorously by CHANDRASEKHAR *et al.* [1958b] and SAGDEYEV *et al.* [1958]. It is due to a concentration of matter in the weak field regions, which produces a transverse magnetic field expansion. The latter increases the mirror ratio in the same regions, and this enhances in its turn the concentration of matter.

We finally observe that the group velocity $U_g = \partial\omega/\partial\kappa$ has components only along the magnetic field for the waves (6.52) and (6.57). For the solutions (6.53) and (6.58) it also permits propagation in certain other directions.

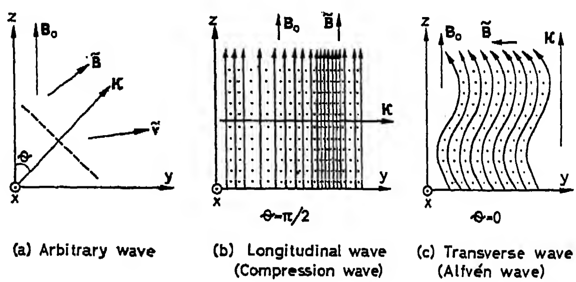


Fig. 6.6. Plane magnetohydrodynamic wave of wave number κ moving in a homogeneous external magnetic field $\boldsymbol{B}_0$. (a) Arbitrary direction of wave normal. (b) Longitudinal wave with $\theta = \frac{1}{2}\pi$. (c) Transverse wave with $\theta = 0$.

CHAPTER 7

CONFINEMENT OF CHARGED PARTICLES

It is clear from the discussions in the preceding chapters that ionized matter is limited in its motion across a strong magnetic field. Figuratively speaking, it should be "frozen" more or less effectively to the magnetic lines of force. The general behaviour of an ionized gas in a magnetic field is essentially based upon this feature.

The present chapter will be devoted to a special study of the confinement of charged particles in different types of magnetic field configurations. Among the astrophysical applications of this subject may be mentioned the study of forbidden regions for cosmic rays in the terrestrial field and of the mechanisms which govern the Van Allen belts. Another important application concerns the study and design of "magnetic bottles" for the confinement and heating of a dense plasma at thermonuclear temperatures. Such temperatures are of the order of 10^8 °K. They can be reached by means of practically available amounts of energy only if the particle losses out of the confinement region are reduced to their utmost limits. In addition to these losses energy can also be drained from a thermonuclear system by such effects as momentum losses by viscosity and heat losses by radiation and thermal conduction.

The motion and confinement of a particle in certain fields of regular geometry is treated in §§ 1–3 of this chapter. In § 4 a summary will further be given of a number of loss mechanisms by which particles escape the confinement of a "magnetic bottle".

1. Survey of Different Confinement Principles

There are several schemes which have been proposed for the production of a magnetic confinement. As seen from the single particle picture the total velocity $\boldsymbol{w}$ consists of three components, $\boldsymbol{W}$, $\boldsymbol{u}_{\parallel}$ and $\boldsymbol{u}_{\perp}$, which represent a gyration, a longitudinal and a transverse guiding centre drift. The gyration forms by itself a closed orbit in first order and the confinement therefore

mainly concerns the drift motions $\boldsymbol{u}_\parallel$ and $\boldsymbol{u}_\perp$ of the guiding centre. In order to obtain a perfect magnetic bottle we have to limit both these drifts. This can be done either by an arrangement where the particle orbits are closed inside the intended confinement region or by producing turning points for the motion at the boundary of the latter.

For the velocity $\boldsymbol{u}_\parallel$ this implies that the magnetic field lines should not lead from the trapped plasma out to external space, or there should be some force which turns the longitudinal motion back into the plasma. The latter situation can be achieved by means of magnetic mirrors or by a centrifugal force.

For the velocity $\boldsymbol{u}_\perp$ the geometry of the applied magnetic field and other force fields can be chosen in such a way that the transverse drift orbits do not lead out to external space. There also exists the possibility of producing turning points for $\boldsymbol{u}_\perp$, e.g., by the application of oscillating fields. For a cycle of motion around the configuration it is also possible to cancel the net transverse displacements by means of a "rotational transform".

In the present discussions on single particle motion it is immaterial whether part of the magnetic and electric fields arises from currents and space charges in the plasma itself or not. This is, of course, only so as long as we take neither the conditions for a steady-state equilibrium into account, nor the conditions for instabilities to develop. If the state of the plasma changes, also the self-fields generated by the plasma currents and space charges will change, and this affects in its turn the resulting magnetic field and its confining properties.

Table 7.1 gives a summary of some of the confinement types which have been suggested and may serve as an illustration to the present survey.

Before turning to a detailed analysis of magnetic bottles it should be mentioned that an ionized gas can possibly be trapped by other means than by a magnetic field. Thus, a number of methods for radio-frequency confinement have been investigated by VEDENOV *et al.* [1958], LINHART [1960] and many others. Even if a trapping of this kind hardly becomes complete it may be combined with a magnetic bottle to block the leaks of the latter. Recent experiments by ARSENIEV *et al.* [1962] have shown that the particle losses along the magnetic field lines are reduced by means of an applied radiation pressure. These confinement methods can therefore be used to suppress the energy loss from escaping particles, but it should also be remembered that they require additional energy to be fed into the system to produce the radiation pressure.

TABLE 7.1.

A summary of proposed confinement schemes. For axially symmetric configurations a poloidal field is situated in planes through the axis of symmetry and a toroidal field is perpendicular to the same planes. The present survey does not concern the stability of the proposed confinement types.

Type of confinement	Direction of magnetic field lines	Longitudinal drift $\boldsymbol{u}_{\parallel}$	Transverse drift $\boldsymbol{u}_{\perp}$	Imposed or external magnetic field	Magnetic self-field
Toroidal pinch	Closed inside plasma	Closed inside plasma	Towards walls; partly suppressed by rotational transform from self-field	Imposed toroidal field, additional fields from induced wall currents	Helical
Stellarator	Closed inside plasma	Closed inside plasma	Towards walls, but is suppressed by rotational transform	Helical	Helical
Ringcurrent without leads (Levitron) or with magnetically screened leads	Closed inside plasma, except at zero points of magnetically screened leads	Closed inside plasma	Closed inside plasma except at magnetically screened leads	Helical	Helical
Astron Plasma Betatron	Closed inside plasma	Closed inside plasma	Closed inside plasma	Poloidal; partly produced by relativistic electron stream	Poloidal
Mirror device Cusped and Picket Fence geometries	Towards walls	Towards walls; suppressed by mirror force	Closed inside plasma	Poloidal	Poloidal
Rotating plasma with large radial ratios	Towards walls	Towards walls; suppressed by centrifugal and mirror forces	Closed inside plasma	Poloidal	Poloidal

2. Forbidden Regions

The exact equations of motion can be used to draw some general conclusions about the confinement in magnetic fields with a high degree of symmetry. For such fields it is possible to determine the forbidden regions of a particle even without knowing the detailed solution of its orbit. We have already seen three examples of this in Ch. 2, § 4.2 (ii) and (iii) and § 4.3, where such regions were discussed for the magnetic dipole field, for the field from a line current and for the hyperbolic field.

In this paragraph we shall concentrate mainly on axially symmetric fields which are *poloidal* or *toroidal*. The former are situated in planes through the axis of symmetry, somewhat like the main part of the earth's magnetic field which runs through the earth's poles. The latter are perpendicular to the same planes. The contents of the coming paragraphs is partly based on earlier results by STÖRMER [1955], COSSLETT [1950], LÜST and SCHLÜTER [1957], LEHNERT [1958a, 1959, 1960], FISSER and KIPPENHAHN [1959], HERTWECK [1959] and BONNEVIER and LEHNERT [1960].

2.1. SYMMETRIC CONFIGURATIONS

Introduce the canonical coordinates q_k and momenta p_k defined in Ch. 2, § 3. We also define the line elements $\mathrm{d}l_k$ and scale factors $h_k = h_k(q_k)$ associated with q_k:

$$\mathrm{d}l_k = h_k \mathrm{d}q_k, \quad w_k = \mathrm{d}l_k/\mathrm{d}t = h_k \dot{q}_k. \tag{7.1}$$

For a rectangular system $q_k = (x, y, z)$ we have $h_k = 1$, and for cylindrical coordinates $q_k = (r, \varphi, z)$ the scale factors become $h_r = 1$, $h_\varphi = r$, $h_z = 1$. Assume q_k to be orthogonal.

The Lagrangian (2.47) is with this notation

$$L = \tfrac{1}{2}m \sum_k (h_k \dot{q}_k)^2 - m\phi_g - q\phi + q \sum_k (h_k \dot{q}_k A_k). \tag{7.2}$$

According to (2.51) the generalized momenta are

$$p_k = m h_k^2 \dot{q}_k + q h_k A_k. \tag{7.3}$$

These relations can be inserted into the definition (2.52) of the Hamiltonian, which reduces to

$$H = \tfrac{1}{2}m \sum_k (h_k \dot{q}_k)^2 + m\phi_g + q\phi \tag{7.4}$$

in analogy with equation (2.62). We use (7.3) to eliminate $\dot{q}_k$ and obtain the Hamiltonian

$$H = \frac{1}{2m}\sum_k \frac{(p_k - qh_kA_k)^2}{h_k^{\,2}} + m\phi_g + q\phi = H(q_k, p_k, t). \tag{7.5}$$

Thus, (q_k, p_k, t) can be used as independent variables as pointed out in Ch. 2, § 3.1.

Now suppose that the potentials $\boldsymbol{A}$, ϕ and ϕ_g of the external fields and the scale factors h_k are independent of a certain coordinate q_j. Such a coordinate is said to be cyclic, or ignorable. Examples of this are given by a cylindrically symmetric configuration where $q_j = \varphi$, and by a two-dimensional case where $q_j = z$ and the field quantities do not vary along the z axis. Then, the Hamiltonian (7.5) does not contain q_j explicitly and with $\chi = p_j$ we immediately obtain from equation (2.59)

$$\frac{dp_j}{dt} = 0, \quad p_j = \text{const.} = p_{j0} = mh_{j0}^2\,\dot{q}_{j0} + qh_{j0}A_{j0}, \tag{7.6}$$

where p_{j0} is the generalized momentum of the particle at the starting point (q_{m0}, q_{j0}, q_{n0}) and initial time t_0, and q_m, q_n denote the two residual coordinates which are perpendicular to q_j.

In the expression (7.5) for the Hamiltonian we introduce the result (7.6). Further solve for the two velocities w_m and w_n which do not correspond to the ignorable coordinate q_j:

$$\begin{aligned}\phi_{eq} \equiv \tfrac{1}{2}m(w_m^2 + w_n^2) = H - H_0 + \tfrac{1}{2}mw_0^2 + m(\phi_{g0} - \phi_g) + q(\phi_0 - \phi)\\ - (1/2mh_j^2)\,[mh_{j0}w_{j0} + q(h_{j0}A_{j0} - h_jA_j)]^2 \geqslant 0. \end{aligned}\tag{7.7}$$

As seen from (7.4) this result does not contain A_m and A_n explicitly. The quantity ϕ_{eq} is equal to the kinetic energy of the motion in the mn plane and should always be positive. We shall see later that it can be regarded as an equivalent potential for the motion in this plane.

We now turn to a physical analysis of the obtained results. Let us assume that the vector potential $\boldsymbol{A}$ only has a component A_j in the direction of the cyclic coordinate q_j. This is, e.g., the case for a poloidal field with z along the axis of symmetry; φ then becomes cyclic. Another example is given by the field from an assembly of parallel line currents, where the coordinate z along the currents becomes cyclic. The magnetic flux enclosed by a contour C with line elements along A_j as those in Figures 7.1a and b is

$$\Phi = \iint \hat{\boldsymbol{n}} \cdot \boldsymbol{B}\,\mathrm{d}s = \oint_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = \oint_C A_j h_j \mathrm{d}q_j. \tag{7.8}$$

For such contours the flux will remain constant when the integrand of (7.8) is constant. The relation

$$A_j h_j = \text{const.} \tag{7.9}$$

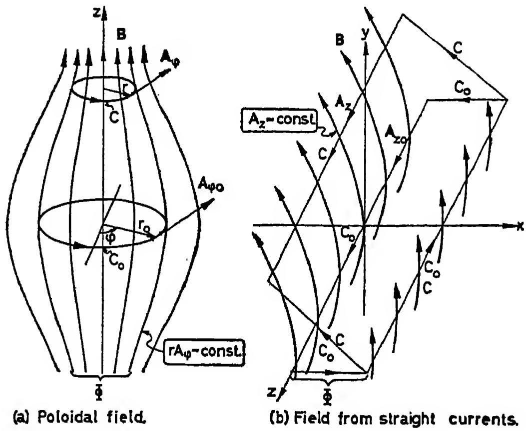


Fig. 7.1. The magnetic field lines form the boundary of regions enclosing the same magnetic flux Φ. (a) Cylindrically symmetric case with poloidal magnetic field. The integral of equation (7.8) has equal values for two contours C_0 and C touching the same field lines. (b) Two-dimensional case with field produced by a straight current system along the z axis. The integral (7.8) has equal values for the two contours C_0 and C.

then describes a magnetic field line in the mn plane. Figure 7.1a shows this for a poloidal field and Figure 7.1b for a straight current configuration, where the field lines are given by $rA_\varphi = \text{const.}$ and $A_z = \text{const.}$, respectively. A toroidal field is a special case of the field in Figure 7.1b where the straight currents are axially symmetric around the z axis and the magnetic field lines form circles in the xy plane around the same axis.

The equivalent potential ϕ_{eq} of equation (7.7) cannot have negative values. Consequently, the forbidden regions for a particle starting at a certain point (q_{m0}, q_{j0}, q_{n0}) is determined by such regions of space where the third member of (7.7) becomes negative. This will certainly be the case when the last term within the square bracket dominates over all other terms of the same member.

Now compare the contribution qh_jA_j from this term with $mh_{j0}w_{j0}$ and with the rest of the terms outside the square bracket of equation (7.7). When the magnetic field is strong enough the Larmor radius becomes very small compared to the characteristic length $L_{cB} = A/|\text{ curl } \boldsymbol{A}|$ along which the magnetic field $\boldsymbol{B} = \text{curl } \boldsymbol{A}$ changes appreciably. The period t_g of gyration

is then also small compared to ϕ/BL^2_{cB} and to $m\phi_g/eBL^2_{cB}$. The latter expressions give the times during which the magnetic field changes noticeably as seen from a frame which follows the electric and gravitation drifts $\boldsymbol{u}_E$ and $\boldsymbol{u}_g = - m\nabla\phi_g \times \boldsymbol{B}/qB^2$ of the particle. For a strong magnetic field each of the terms $h_{j0}A_{j0}$ and h_jA_j in (7.7) will therefore give contributions which are much larger than those of any other term in the same equation. Consequently, the slightest deviation of the particle orbit from the surface $h_jA_j = h_{j0}A_{j0}$ will make the third member of (7.7) strongly negative, i.e., this deviation would displace the particle into a forbidden region. We therefore conclude that the particle will be trapped in a narrow strip around the field line $h_jA_j = h_{j0}A_{j0}$ in the mn plane; the breadth of this region is of the order of the Larmor radius.

It should be observed that this result has been obtained for a time-dependent field, where $\mathrm{d}H/\mathrm{d}t = \partial H/\partial t$ according to Ch. 2, § 3.2. In a stationary state the lines $h_jA_j = h_{j0}A_{j0}$ will not move in space and their projection in the mn plane will run through the starting point. If the field instead changes in time the surfaces $h_jA_j(t) = h_{j0}A_{j0}$ can still be defined by field lines but the latter will then "move" in space. What has been proved for the present symmetric configurations is therefore that the particle follows the moving field lines in such a way that the flux of (7.8) is preserved. This is nothing but a magnetic compression phenomenon where the field lines act like pistons which push the particles, as is also expected from Ch. 4, § 5 and Ch. 6, § 1.

The results (7.6) and (7.3) are easily understood in the axially symmetric case from some simple physical arguments. In the laboratory frame the electric and magnetic fields are

$$\boldsymbol{E} = -\left(\frac{\partial\phi}{\partial r} + \frac{\partial A_r}{\partial t}, \frac{\partial A_\varphi}{\partial t}, \frac{\partial\phi}{\partial z} + \frac{\partial A_z}{\partial t}\right) \tag{7.10}$$

and

$$\boldsymbol{B} = \left(-\frac{\partial A_\varphi}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi)\right). \tag{7.11}$$

In the coordinate system of the particle the electric field becomes $\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{w} \times \boldsymbol{B}$. In this frame we measure a torque

$$qrE'_\varphi = -\left(\frac{\partial}{\partial t} + w_r\frac{\partial}{\partial r} + w_z\frac{\partial}{\partial z}\right)(rA_\varphi) = -\frac{\mathrm{d}}{\mathrm{d}t}(rA_\varphi) \tag{7.12}$$

on the particle with respect to the axis of symmetry. It should be balanced by

the time derivative of the mechanical angular momentum, mrw_φ. This yields equation (7.6) with $p_\varphi = mrw_\varphi + qrA_\varphi$.

Finally the following examples are used to illustrate the stationary state where $H = H_0$:

(i) In a poloidal field, like that given by Figure 7.1a, the equivalent potential becomes

$$\phi_{eq} = \tfrac{1}{2}mw_0^2 + m(\phi_{g0} - \phi_g) + q(\phi_0 - \phi) \\ - (1/2mr^2)\left[mr_0 w_{\varphi 0} + q(r_0 A_{\varphi 0} - rA_\varphi)\right]^2 \geqslant 0. \qquad (7.13)$$

Since ϕ_{eq} cannot become negative the forbidden regions in a strong magnetic field are at least situated outside a narrow strip around the field line $rA_\varphi = r_0 A_{\varphi 0}$ in the rz plane. This is a sufficient condition for the forbidden regions, but not always a necessary one. We shall see later in § 2.2 that the regions may extend further than what can immediately be judged from equation (7.13). Notice that condition (7.13) is not affected by a superimposed toroidal field $A_z = A_z(r)$.

If the magnetic field is purely poloidal as in Figure 7.1a we have $\boldsymbol{A} = (0, A_\varphi, 0)$ and the equation of motion in the rz plane becomes

$$m\ddot{p} = -\nabla\phi_{eq} \qquad (7.14)$$

as is easily verified by direct application of the equation (2.36) of motion. Obviously the particle moves in an equivalent potential "trough" given by ϕ_{eq} and centered closely to the field line through the starting point, (r_0, φ_0, z_0).

An interesting special case is that where the magnetic field is homogeneous, $\phi_g = 0$, and an electric field is applied in the radial direction. Suppose the latter to be large enough for the initial velocity $\boldsymbol{w}_0$ to become unimportant. This corresponds to the conditions of the magnetron tube. Then an electron which starts at the surface $r = r_0$ will not be able to reach a surface $r = r_1$ if the magnetic field $B_0 = 2A_\varphi/r$ satisfies the condition

$$B_0^2 > 8m(\phi_1 - \phi_0)/er_1^2\left[1 - (r_0/r_1)^2\right]^2, \qquad (7.15)$$

where $\phi_1 > \phi_0$ and $r_1 > r_0$. Equation (7.15) is the condition for cut-off of the electron current in the magnetron.

(ii) In a field generated by an assembly of straight currents like that shown in Figure 7.1b the equivalent potential is

$$\phi_{\mathrm{eq}} = \tfrac{1}{2}mw_0^2 + m(\phi_{g0} - \phi_g) + q(\phi_0 - \phi)$$
$$- (1/2m)\left[mw_{z0} + q(A_{z0} - A_z)\right]^2 \geqslant 0. \qquad (7.16)$$

The motion in the xy plane obeys an equation analogous to equation (7.14).

For a toroidal field we have $\boldsymbol{A} = (0, 0, A_z)$, where $A_z \propto \log(1/r)$. The particles are then confined to cylindrical shells centered around the axis of symmetry and having a thickness of the order of the Larmor radius. This is, of course, only true as long as the assumption $\partial H/\partial z = 0$ holds. In the later parts of the present chapter we shall discuss the consequences of a charge separation in the axial direction and the corresponding changes in the electric potential ϕ. An axial electric field $-\partial\phi/\partial z$ then arises and the Hamiltonian is no longer independent of z.

For two current leads immersed in an external homogeneous magnetic field B_0 in the y direction the resulting field is as shown in Figure 7.2. The distance between the leads is $2d_1$. With the currents having the directions given in the figure and the magnitudes $I_1 = \pm\, 4\pi d_1 B_0/\mu_0$, the leads become force-free and are surrounded by a region in which the field lines circulate around their centra. The region is bounded by the surface S corresponding to a certain constant value of A_z. Consider a particle which approaches the

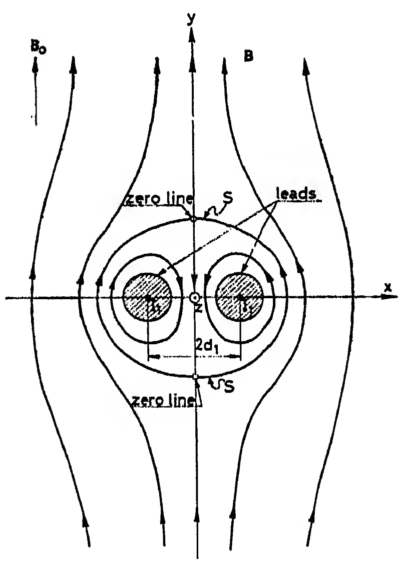


Fig. 7.2. A pair of force-free current leads infinitely extended in the z direction and immersed in a homogeneous magnetic field $\boldsymbol{B}_0$ (LEHNERT [1959]).

lead region from infinity and from any direction. It is then immediately clear from (7.16) that the particle will not be able to reach the surfaces of the leads if the magnetic field is strong enough. The reason for this is that the values of qA_z at a lead surface deviate strongly from that at S.

2.2. ROTATING SYSTEMS

In the preceding paragraph has been shown how a particle can be confined to the magnetic field lines under certain circumstances. It can also be trapped in the longitudinal direction, e.g. between two magnetic mirrors. This is possible when its velocity components are related in a suitable way, as has already been demonstrated in Chapter 6. Here we shall consider another possibility of reducing the longitudinal particle losses. It is due to an artificial "gravitation" force which arises from the centrifugal acceleration in a rotating system. This type of confinement was first suggested by BAKER and ANDERSON [1956] and has been further treated by ANDERSON *et al.* [1958], BOYER *et al.* [1958], LONGMIRE *et al.* [1959], BONNEVIER and LEHNERT [1960] and LEHNERT [1960, 1962b].

Consider a particle the position vector of which is $\boldsymbol{\rho} = \boldsymbol{\rho}(t)$ in the laboratory system. Introduce in this system three orthogonal vectors, $\boldsymbol{x}^*(t)$, $\boldsymbol{y}^*(t)$, $\boldsymbol{z}^*(t)$, the sum of which is equal to the position vector at any time:

$$\boldsymbol{\rho}(t) = x^*\hat{\boldsymbol{x}}^* + y^*\hat{\boldsymbol{y}}^* + z^*\hat{\boldsymbol{z}}^* \equiv \boldsymbol{\rho}^*. \tag{7.17}$$

Here $(\hat{\boldsymbol{x}}^*, \hat{\boldsymbol{y}}^*, \hat{\boldsymbol{z}}^*)$ are time-dependent unit vectors. The velocity of the particle is then

$$\boldsymbol{w} = \dot{\boldsymbol{\rho}} = \dot{x}^*\hat{\boldsymbol{x}}^* + \dot{y}^*\hat{\boldsymbol{y}}^* + \dot{z}^*\hat{\boldsymbol{z}}^* + x^*\dot{\hat{\boldsymbol{x}}}^* + y^*\dot{\hat{\boldsymbol{y}}}^* + z^*\dot{\hat{\boldsymbol{z}}}^*, \tag{7.18}$$

with a dot indicating time derivative. In the same way the acceleration becomes

$$\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t} = \ddot{\boldsymbol{\rho}} = \ddot{x}^*\hat{\boldsymbol{x}}^* + \ddot{y}^*\hat{\boldsymbol{y}}^* + \ddot{z}^*\hat{\boldsymbol{z}}^* + 2(\dot{x}^*\dot{\hat{\boldsymbol{x}}} + \dot{y}^*\dot{\hat{\boldsymbol{y}}}^* + \dot{z}^*\dot{\hat{\boldsymbol{z}}}^*)$$
$$+ x^*\ddot{\hat{\boldsymbol{x}}}^* + y^*\ddot{\hat{\boldsymbol{y}}}^* + z^*\ddot{\hat{\boldsymbol{z}}}^*. \tag{7.19}$$

Now specify $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{z}^*)$ by stating that $\boldsymbol{z}^* = z\hat{\boldsymbol{z}}$ and that

$$\dot{\hat{\boldsymbol{x}}}^* = \boldsymbol{\Omega} \times \hat{\boldsymbol{x}}^*, \quad \dot{\hat{\boldsymbol{y}}}^* = \boldsymbol{\Omega} \times \hat{\boldsymbol{y}}^*, \tag{7.20}$$

and

$$\ddot{\hat{\boldsymbol{x}}}^* = \dot{\boldsymbol{\Omega}} \times \hat{\boldsymbol{x}}^* + \boldsymbol{\Omega} \times \dot{\hat{\boldsymbol{x}}}^*, \quad \ddot{\hat{\boldsymbol{y}}}^* = \dot{\boldsymbol{\Omega}} \times \hat{\boldsymbol{y}}^* + \boldsymbol{\Omega} \times \dot{\hat{\boldsymbol{y}}}^*, \tag{7.21}$$

where $\boldsymbol{\Omega} = \hat{\boldsymbol{z}}\Omega$ is a function of time and of $(x^{*2} + y^{*2})$. This implies that the vectors $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ rotate at a time dependent angular velocity $\boldsymbol{\Omega}$ which is a function of the distance from the axis $\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}^*$ of rotation. We can then interpret the vectors $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{z}^*)$ as the coordinate axes of a frame rotating at the local and instantaneous angular velocity $\boldsymbol{\Omega}$ around $\hat{\boldsymbol{z}}$, and $\boldsymbol{\rho}^*$ as the position vector in this new frame. The velocity and acceleration measured in the same frame are

$$\boldsymbol{w}^* = \dot{x}^*\hat{\boldsymbol{x}}^* + \dot{y}^*\hat{\boldsymbol{y}}^* + \dot{z}^*\hat{\boldsymbol{z}}^*, \quad \frac{\mathrm{d}\boldsymbol{w}^*}{\mathrm{d}t} = \ddot{x}^*\hat{\boldsymbol{x}}^* + \ddot{y}^*\hat{\boldsymbol{y}}^* + \ddot{z}^*\hat{\boldsymbol{z}}^*. \tag{7.22}$$

Combination of equations (7.17) – (7.22) yields

$$\boldsymbol{w} = \boldsymbol{w}^* + \boldsymbol{\Omega} \times \boldsymbol{\rho}^* \tag{7.23}$$

for the velocity, and

$$\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{w}^*}{\mathrm{d}t} + 2\boldsymbol{\Omega} \times \boldsymbol{w}^* - \boldsymbol{\Omega} \times (\boldsymbol{\rho}^* \times \boldsymbol{\Omega}) + \frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t} \times \boldsymbol{\rho}^* \tag{7.24}$$

for the absolute acceleration. The second term of the right hand member of (7.24) is the Coriolis force and the third term is the centrifugal force. The equation of motion is

$$m\frac{\mathrm{d}\boldsymbol{w}^*}{\mathrm{d}t} = q\boldsymbol{E} + q\boldsymbol{w}^* \times \boldsymbol{B} + q(\boldsymbol{\Omega} \times \boldsymbol{\rho}^*) \times \boldsymbol{B} + 2m\boldsymbol{w}^* \times \boldsymbol{\Omega}$$

$$- m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{\rho}^*) + m\boldsymbol{\rho}^* \times \frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t} - m\nabla\phi_{\mathrm{g}} \tag{7.25}$$

in the rotating system (see also CHANDRASEKHAR [1961]).

In the axially symmetric case where all quantities are independent of the coordinate φ of a system (r, φ, z) the electric and magnetic fields are given by equations (7.10) and (7.11). The φ component of (7.25) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(rw_\varphi^* + \frac{q}{m}\,rA_\varphi\right) + 2\Omega r\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\mathrm{d}\Omega}{\mathrm{d}t}\,r^2 = 0, \tag{7.26}$$

which is integrated to

$$mrw_\varphi^* + qrA_\varphi + m\Omega r^2 = mr_0w_{\varphi 0}^* + qr_0A_{\varphi 0} + m\Omega_0r_0^2 = p_{\varphi 0}. \tag{7.27}$$

Here $p_{\varphi 0}$ is a constant having the form of a generalized angular momentum. This result is equivalent to (7.6) and can also be obtained if $\boldsymbol{w}$ is substituted from (7.23) into (7.6).

We shall now bring (7.25) to a form equivalent to (2.36) by introducing the modified potentials

$$\boldsymbol{A}^* = \boldsymbol{A} + \frac{m}{q}\,\boldsymbol{\Omega} \times \boldsymbol{r}, \quad \boldsymbol{r} = (r, 0, 0) \tag{7.28}$$

and

$$\phi^* = \phi - \tfrac{1}{2}\,\frac{m}{q}\,\Omega^2 r^2 + \Omega(p_{\varphi 0}/q - rA_\varphi). \tag{7.29}$$

They correspond to the modified electric and magnetic fields

$$\boldsymbol{E}^* = -\nabla\phi^* - \frac{\partial \boldsymbol{A}^*}{\partial t} = \boldsymbol{E} + \frac{m}{q}\,\Omega r \nabla(\Omega r) - \nabla(\Omega p_{\varphi 0}/q - \Omega r A_\varphi) + \frac{m}{q}\,\boldsymbol{r} \times \frac{\partial \boldsymbol{\Omega}}{\partial t} \tag{7.30}$$

and

$$\boldsymbol{B}^* = \operatorname{curl} \boldsymbol{A}^* = \boldsymbol{B} + 2\,\frac{m}{q}\,\boldsymbol{\Omega} + \frac{m}{q}\,(\boldsymbol{r} \times \hat{\boldsymbol{\Omega}}) \times \nabla\Omega. \tag{7.31}$$

With these definitions equation (7.25) changes into the form

$$m\,\frac{\mathrm{d}\boldsymbol{w}^*}{\mathrm{d}t} = q\boldsymbol{E}^* + q\boldsymbol{w}^* \times \boldsymbol{B}^* - m\nabla\phi_g \tag{7.32}$$

as is easily seen when expressions (7.30) and (7.31) are inserted into (7.32) and use is made of equations (7.11) and (7.27). The obtained result (7.32) is analogous to the equation of motion (2.36). Of special interest are the second terms of the last members of equations (7.30) and (7.31) which correspond to the centrifugal force and to the Coriolis force, respectively.

Equation (7.32) refers to a coordinate system which follows the rotation. The accelerations arising from the latter, and the corresponding transverse particle drifts, have been taken explicitly into account. When $\boldsymbol{E}^*$ satisfies the conditions of Ch. 3, § 1 we can therefore develop a first order orbit theory on the basis of (7.32) for the motions observed in the rotating frame.

We now return to the discussion on the forbidden regions and consider a stationary state. According to equations (7.32), (7.28) and (7.29) the equivalent potential (7.13) in a rotating frame is given by an expression where $\boldsymbol{A}$, ϕ, $\boldsymbol{w}$ are substituted by $\boldsymbol{A}^*$, ϕ^*, $\boldsymbol{w}^*$. After such a substitution has been made the equivalent potential of (7.13) is changed into the modified form

$$\begin{aligned}\phi^*_{\mathrm{eq}} = \tfrac{1}{2}m(w_r^{*2} + w_z^{*2}) &= \tfrac{1}{2}m w_0^{*2} + m(\phi_{g0} - \phi_g) + q(\phi_0 - \phi) \\ &- q(\Omega_0 r_0 A_{\varphi 0} - \Omega r A_{\varphi 0}) + (\Omega_0 - \Omega)p_{\varphi 0} - \tfrac{1}{2}m(\Omega_0^2 r_0^2 - \Omega^2 r^2) \\ &- (1/2mr^2)\left[m r_0 w_{\varphi 0}^* + q(r_0 A_{\varphi 0} - rA_\varphi) + m\Omega_0 r_0^2 - m\Omega r^2\right]^2 \geqslant 0.\end{aligned} \tag{7.33}$$

The same result can be derived from an insertion of expression (7.23) into (7.13) and from a rearrangement of the obtained terms. It can also be deduced from (7.27) in combination with the condition (2.38) for conservation of energy.

So far the angular velocity $\boldsymbol{\Omega}$ has not been specified. Now consider an ionized gas immersed in the poloidal field of Figure 7.3. We restrict the discussion to a situation where the longitudinal and transverse particle energies are comparable and where the latter corresponds to a potential which is much smaller than the transverse potential difference across the whole macroscopic configuration. Since the particles can move quite easily along the magnetic field lines it is then justified to suppose that the applied electric field $\boldsymbol{E}$ has a longitudinal component $\boldsymbol{E}_{\parallel}$ which is small compared to the transverse component $\boldsymbol{E}_{\perp}$. In other words, we assume $\boldsymbol{E}_{\parallel}$ to be of first order in the smallness parameter, whereas $\boldsymbol{E}_{\perp}$ is now allowed to be of zero order. This is consistent with the approximation (3.34) to lowest order. We therefore assume that the electric field adjusts itself so that the magnetic field lines nearly become equipotentials. In a general case this requires some space charge formation. For a narrow cylindrical shell between two field lines with the electric potential difference $\mathrm{d}\phi$ the electric field is $E = -\mathrm{d}\phi/\mathrm{d}l_{\perp}$ with the directions given in the figure. In lowest order all charged particles

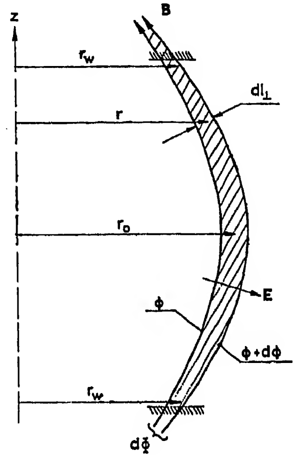


Fig. 7.3. The shaded area in the figure represents the cross section of a thin toroidal shell generated by the field lines of a strong poloidal field $\boldsymbol{B}$. An electric field $\boldsymbol{E}$ is imposed at right angles to $\boldsymbol{B}$.

move with the electric field drift. The whole mass of gas then rotates around the axis of symmetry at the angular velocity

$$\boldsymbol{\Omega} = \boldsymbol{E} \times \boldsymbol{B}/rB^2. \tag{7.34}$$

Use this relation as a definition of $\boldsymbol{\Omega}$. It implies that we separate the electric field drift from the rest of the particle motion and introduce a frame of reference where the former disappears. The magnetic flux $\mathrm{d}\Phi = 2\pi rB\mathrm{d}l_\perp$ enclosed by the strip is related to $\mathrm{d}\phi$ and to Ω by

$$\Omega = -E/Br = 2\pi\mathrm{d}\phi/\mathrm{d}\Phi = \mathrm{d}\phi/\mathrm{d}(rA_\varphi). \tag{7.35}$$

It should be observed that both $\mathrm{d}\phi$ and $\mathrm{d}\Phi$ are constant along the strip, and that the gas therefore will rotate at the constant angular velocity $\Omega = \Omega_0$ along the entire length of a field line. This is consistent with the isorotation law earlier discovered by FERRARO [1937].

It is now assumed that the magnetic field is very strong so that each of the contributions from $r_0A_{\varphi 0}$ and rA_φ in (7.33) is much greater than that of any other term in the same equation. This is justified under the same conditions which were already discussed in connexion with equation (7.7). It also implies that the angular velocity Ω should be much smaller than the gyro frequency ω_g. We can then still draw the conclusion that the particle is confined inside a narrow shell around $rA_\varphi = \text{const.}$, as that shown in Figure 7.3. However, with the present representation we have split off the rotation by means of the transformation (7.23). As a consequence, more information can be gained about the forbidden regions than what is directly obtained from equation (7.13). Since the particle has to move inside the shell, Ω will have values close to Ω_0 all along the particle orbit. According to (7.35) the third and fourth terms of the third member of expression (7.33) then cancel in the limit of infinitely strong magnetic fields. In this limit we arrive at the expression

$$\begin{aligned}\phi_{\mathrm{eq}}{}^* = {} & \tfrac{1}{2}mw_0{}^{*2} - \tfrac{1}{2}m\Omega_0^2(r_0^2 - r^2) \\ & - (1/2mr^2)\left[mr_0w_{\varphi 0}{}^* + q(r_0A_{\varphi 0} - rA_\varphi) + m\Omega_0(r_0^2 - r^2)\right]^2 \geqslant 0\end{aligned} \tag{7.36}$$

when the gravitation field is neglected. The second term in the second member of this expression represents the work performed by the particle against the centrifugal field. The last term within the square bracket arises from the Coriolis force. A condition analogous to equations (7.7) and (7.33) for a time dependent external field can also be derived for a rotating system where $\Omega = \Omega(t)$. A detailed analysis will not be given here.

Relation (7.36) shows that there is absolute confinement for a particle inside a region $r > r_w$ of Figure 7.3 provided that the thermal energy $\frac{1}{2}mw_0^{*2}$ at the starting point r_0 is less than the centrifugal work $\frac{1}{2}m\Omega_0^2(r_0^2 - r_w^2)$. The latter approaches the value $\frac{1}{2}m\Omega_0^2 r_0^2$ for large ratios r_0/r_w.

When $\Omega \ll \omega_g$ the modification introduced by the Coriolis force inside the square bracket of (7.36) becomes small. It affects the shape of the "bottom line" in the "potential trough" corresponding to $\phi_{eq}{}^*$. The reason for this follows from (7.31) which shows that the particle moves in an equivalent magnetic field $\boldsymbol{B}^*$ which differs slightly from the field $\boldsymbol{B}$ observed in the laboratory frame. Thus the particle will no longer "slide" exactly along the magnetic field lines in the limit of an infinitely small Larmor radius. It will instead be slightly deflected from these lines due to the Coriolis force. This effect depends upon m/q. Even if it is of minor importance to the present analysis we shall see later in Ch. 8, § 2.7 that it may have influence on charge separation phenomena.

3. Confinement in Different Field Configurations

A wide variety of field configurations and methods have been suggested for the confinement of a high-temperature plasma. In this paragraph we shall discuss some concrete examples on the basis of the theories on forbidden regions and particle orbits.

3.1. POLOIDAL FIELDS

The magnetic *mirror* devices investigated by POST [1958], COENSGEN *et al.* [1958] and many others are in principle as sketched in Figure 6.3. Plasma is contained between two mirrors at $s = s_1$ and $s = s_2$ and is compressed as described in Ch. 6, § 2.1.

The mechanism for mirror reflection is easily treated in terms of the longitudinal equations of motion (3.17) and (4.6) of the orbit theory. Since the magnetic moment is a constant of the motion we have

$$m\frac{\mathrm{d}u_{\parallel}}{\mathrm{d}t} \approx -q\frac{\partial\phi}{\partial s} - \frac{\partial}{\partial s}(MB), \tag{7.37}$$

where s is the coordinate along the magnetic field. Clearly $q\phi + MB$ plays the rôle of a potential in this problem. Assume a quasi-stationary state. During a period $t_{\parallel}$ of the longitudinal motion between the mirrors the particle moves along the surface $rA_\varphi = r_0 A_{\varphi 0}$ and ϕ and B are only functions

of s, in the first approximation. We can then multiply equation (7.37) by $u_{\|}$ and integrate with respect to s. From this follows that

$$\tfrac{1}{2}m(u_{\|}^2 - u_{\|0}^2) = q(\phi_0 - \phi) - M(B - B_0). \tag{7.38}$$

We now introduce the angle θ between the total velocity $\boldsymbol{w}_0$ and the longitudinal component $\boldsymbol{u}_{\|0}$ at the starting point. Due to the smallness of the drift $\boldsymbol{u}_{\perp}$ we can put $w_{0\perp}^2 \approx W_0^2$, where $\boldsymbol{W}_0$ is the corresponding velocity of gyration. For the turning points $s_1 = -s_m$ and $s_2 = s_m$ at the mirrors of a symmetric configuration as in Figure 6.3 we have $u_{\|} = 0$. Then,

$$\tfrac{1}{2}mu_{\|0}^2 = \tfrac{1}{2}mW_0^2/(\mathrm{tg}\,\theta)^2 = q(\phi_m - \phi_0) + \tfrac{1}{2}mW_0^2(R_m - 1), \tag{7.39}$$

where subscripts ($_0$) and ($_m$) refer to the starting point in the equatorial plane of Figure 6.3 and to the mirror points, respectively. Further the mirror ratio $R_m = B_m/B_0$ of (6.20) has been introduced. It should have the value

$$R_m = (\sin\theta)^{-2} + 2q(\phi_0 - \phi_m)/mW_0^2 \tag{7.40}$$

for the particle to be reflected.

In absence of electric fields all particles inside the cones given by $(\sin\theta)^2 > > 1/R_m$ will remain trapped between the mirrors according to the present approximate theory. On the other hand, particles for which $(\sin\theta)^2 < 1/R_m$ will be lost. The corresponding values $\theta_m = \arcsin(1/R_m)^{\frac{1}{2}}$ define two *loss cones* in the directions parallel and antiparallel with the longitudinal component w_s in velocity space. This is indicated in Figure 7.4. Starting from

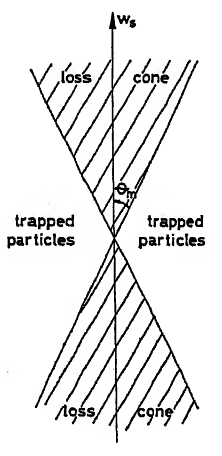


Fig 7.4. The loss cones of a mirror device in velocity space.

some mathematical discoveries of MOSER [1962], GARDNER [1962] has recently shown that if the magnetic field is strong enough a particle outside these loss cones will be contained for all times.

If an electric field is present the mirror ratio for particle reflection is no longer proportional to $(\sin \theta)^{-2}$, and the presence of ϕ in equation (7.40) makes the loss cones depend upon $\sin \theta$. For $q(\phi_0 - \phi_m) > 0$ electrostatic forces will act in a way to pull out the particle through the mirrors and a larger mirror ratio R_m is required to produce a turning point for $\boldsymbol{u}_\parallel$. The electric field may arise from ambipolar effects as discussed in § 4.4.

We further conclude from equation (7.38) that it is possible for the potential difference $\phi - \phi_0$ along a magnetic field line to approach the value $-(mW_0^2/2q)(R-1)$ and still have a particle trapped between the magnetic mirrors. If the ratio $R = B/B_0$ is far greater than unity and if $u_{\parallel 0}$ and W_0 are of the same magnitude, this implies that $q(\phi_0 - \phi)$ may become much greater than $\frac{1}{2}mW_0^2$ for such a particle. This result, which was first obtained by ALFVÉN and FÄLTHAMMAR [1963], is at least valid as long as conditions (3.1) and (3.2) of the perturbation theory are satisfied. It is also consistent with the second of expressions (3.34) since the magnetic gradient term in (7.37) is of order ε according to Chapter 3, § 1.3.

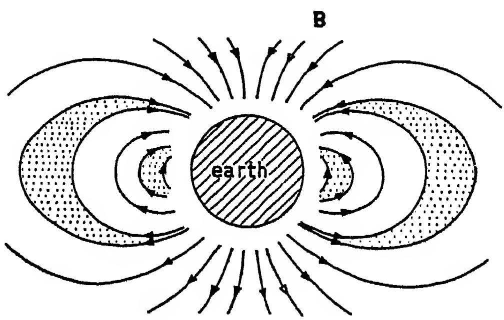


Fig. 7.5. Sketch of the Van Allen belts of particles trapped in the earth's magnetic field.

We observe that when $q(\phi_0 - \phi) > 0$ the longitudinal electric field forces the particle into a stronger magnetic field, provided that the latter has a shape as indicated in Figure 6.3. The increase in the Larmor energy from $\frac{1}{2}mW_0^2$ to $\frac{1}{2}mW^2$ can then be considered as the result of a transverse magnetic compression which takes its energy both from the longitudinal velocity $\boldsymbol{u}_\parallel$ and from the longitudinal electric field $\boldsymbol{E}_\parallel$.

The poloidal field used in a mirror device has the advantage that the

transverse drift $\boldsymbol{u}_\perp$ of particles given by (3.22) forms closed orbits in the φ direction. In an equilibrium state no charge separation is therefore generated by the transverse drifts.

There exists at least one cosmic example of magnetic mirror confinement in a nearly symmetric configuration. This is given by the belts of particles trapped in the earth's magnetic field, as discovered by VAN ALLEN [1959] and sketched in Figure 7.5.

The configuration of this figure has the advantage of large mirror ratios. One can proceed one step further in order to reduce the end losses from a mirror device of similar geometry. This is shown in Figure 7.6 where a

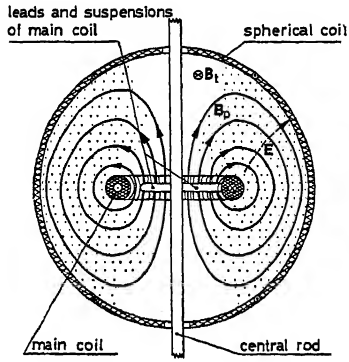


Fig. 7.6. Magnetic bottle with poloidal field $\boldsymbol{B}_\text{p}$ formed by a ring current through the main coil in the figure in combination with the field from a surrounding spherical coil. The main coil is supported by leads. The plasma trapped in the magnetic field can be set into rotation around the axis of symmetry by an applied transverse electric field $\boldsymbol{E}$. A toroidal magnetic field $\boldsymbol{B}_\text{t}$ may be superimposed for stabilizing purposes (LEHNERT [1958a, 1959, 1960], BONNEVIER and LEHNERT [1960]).

poloidal field $\boldsymbol{B}_\text{p}$ is produced by a ring-shaped coil and a surrounding spherical coil, and a toroidal field $\boldsymbol{B}_\text{t}$ may be superimposed for stabilizing purposes (LEHNERT [1958a, 1959]). Since the main coil is suspended by current leads in the form of spokes, most of the field lines can pass freely through its interior and the cross section for end losses at the mirrors is reduced. An additional reduction is possible when the leads are arranged in a way given by Figure 7.2 which shows their cross section. By such an arrangement the lead surfaces become magnetically screened (cf. also GREYBER [1958]).

As an alternative approach to the ring current configuration of Figure 7.6

the leads can be removed so that the main coil hangs freely in space and all field lines are closed inside the confinement region. The coil position then has to be controlled by some means. One possibility has been realized in the "Levitron" by COLGATE and FURTH [1958] (see also BIRDSALL *et al.* [1960]). Here the core is levitated by induced electric currents and is kept in due position for some time by means of the same currents.

In the "Astron" device by CHRISTOFILOS [1958] a ring current is instead produced by a layer of high energy electrons. Together with an external magnetic mirror field this layer produces a resulting field with closed lines of force inside the confinement region. One of the main problems of this device concerns the stability of the electron layer.

ALFVÉN [1958] has suggested that a ring current may be produced by the plasma itself. The problem is then to shoot a heated plasma into a magnetic field in such a way that a stable plasma ring is formed which is confined in a static equilibrium only by means of its self-field.

The mirror confinement can be enhanced by the help of the centrifugal force if the plasma is set into rotation as proposed by BAKER and ANDERSON [1956] and shown in § 2.2. According to BAKER [1950] a rotating plasma is easily generated by crossed electric and magnetic fields. Such conditions have been established by BOYER *et al.* [1958] in the "Ixion" device where a discharge runs across a magnetic mirror field. Similarly, in the device of Figure 7.6 the trapped plasma can be set into rotation around the axis of symmetry when an electric field is applied between the main coil on one hand, and the spherical coil and the central rod on the other (LEHNERT [1959]). It has earlier been shown BONNEVIER and LEHNERT [1960] by that the particle losses along the magnetic field are controlled by the centrifugal work in the radial direction, and not necessarily by the mirror ratio. The configuration of Figure 7.6 is especially suitable for a strong centrifugal confinement, on account of its large radial extensions. This is also obvious from a discussion in terms of the macroscopic fluid equations (LEHNERT [1960]).

Due to the equivalence between equations (7.32) and (2.36) we can immediately apply the result (7.38) from the orbit theory to a rotating system with the equivalent fields ϕ^* and A^* given by equations (7.28) and (7.29). The modified equivalent magnetic moment $M^* = mW_0^{*2}/2B^*$ should then be a constant of the motion for a particle moving in the modified magnetic field $\boldsymbol{B}^*$ of equation (7.31). This implies that we use a somewhat better approximation to the perturbation theory of Chapter 3 by taking

the regular rotation $\boldsymbol{\Omega}$ into account explicitly. In a stationary state the relation equivalent to (7.38) becomes

$$\tfrac{1}{2}m(u_{\|}{}^{*2} - u_{\|0}{}^{*2}) = q(\phi_0 - \phi) + (\Omega_0 - \Omega)p_{\varphi 0} + q(\Omega r A_\varphi - \Omega_0 r_0 A_{\varphi 0}) - \tfrac{1}{2}m(\Omega_0^2 r_0^2 - \Omega^2 r^2) - M^*(B^* - B_0{}^*). \quad (7.41)$$

When the magnetic field is strong and $\Omega \ll \omega_g$ the corresponding relation for the turning points of the longitudinal motion is

$$\tfrac{1}{2}mu_{\|}{}^{*2} \approx \tfrac{1}{2}mW_0{}^{*2}(R_m - 1) + \tfrac{1}{2}m\Omega_0^2(r_0^2 - r^2). \quad (7.42)$$

We have here assumed that there are no electrostatic fields in the frame which follows the rotation at the points along the field line through r_0, z_0. The present relation shows how the mirror containment due to the mirror ratio R_m is enhanced by the centrifugal containment due to the *radial ratio* r_0/r.

As compared to an ordinary mirror device the configuration of Figure 7.6 has the advantage that strong magnetic fields can be obtained in the "equatorial" plane of the main coil, at the same time as large radial ratios are preserved. A further development of the arrangement in Figure 7.6 along these lines is possible by increasing the dimensions of the main coil with respect to those of the external coil system (the spherical coil). The confinement region will then consist of a thin toroidal shell within which there is a high magnetic field strength in the equatorial plane where the plasma density and the centrifugal force have their maxima. This should improve stability.

In this connexion we should mention that the velocity of rotation in a straight Homopolar device appears to be limited to the value $(2eV_i/m_i)^{\frac{1}{2}}$, where V_i is the ionization potential and m_i the ion mass (Fahleson [1961], Angerth *et al.* [1962]). This occurs within a wide range of discharge parameters and is consistent with a theory by Alfvén [1954b]. The latter suggests that a neutral gas cloud will put a strong brake on a cloud of ionized matter, as soon as the relative velocity of the clouds approaches the value just mentioned. In a rotating plasma device this situation may be realized by the back-flux of neutral gas. It arises from plasma particles which escape along the field lines to the end insulators and recombine there. If this mechanism produces a velocity limit in a stationary state, and if the angular velocity $\boldsymbol{\Omega}$ of rotation is constant along a field line, we see that equatorial velocities can be obtained which exceed the velocity limit and reach the value $(r_0/r_w)\,\cdot$

$\cdot (2eV_i/m_i)^{\frac{1}{2}}$. Here r_0/r_w is the radial ratio with respect to the radius r_w at the walls of the end insulators (cf. Figure 7.3). In a stationary state large velocities of rotation would then be available only in devices with large radial ratios.

So far we have discussed magnetic bottles of the poloidal type where there exist regions in which the magnetic field bends concavely towards the main plasma body. As will be seen later in Chapter 8 this may sometimes produce instabilities and spoil the confinement (BERKOWITZ *et al.* [1958], BERNSTEIN *et al.* [1958]). To avoid such instabilities the magnetic field can instead be arranged such as to bend convexly towards the plasma all over its boundary. This leads to the *cusped* geometry suggested by TUCK [1958] and shown in Figure 7.7. A combination of cusps gives the "Picket Fence" configu-

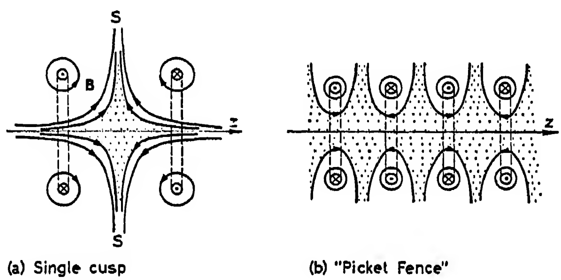


Fig. 7.7. Magnetic bottles with field bending convexly towards the plasma body. (a) Single cusp. (b) "Picket Fence".

ration of Figure 7.7b. In addition to the mirror ends at the axis there exist ring-shaped regions S through which particles can escape.

At the centre of a cusp there is a zero-point around which a weak magnetic field region exists. This region has the disadvantage of acting somewhat like a big scattering centre, because the adiabatic invariance breaks down for particles which pass through it. To avoid the corresponding particle leakage to the loss cones a "stuffed" cusp can be arranged where a central conductor is placed along the axis of symmetry. A current which is fed through the rod generates a superimposed toroidal magnetic field. This removes the particle leakage just mentioned, but introduces at the same time an inner boundary of the confinement region where the magnetic field bends concavely towards the plasma body.

There are several methods by which a plasma can be introduced into a mirror device, as shown in a survey by POST [1958]. By one of them the

plasma is injected into the confinement zone through one of the mirrors. It is then trapped between the mirrors when the magnetic field rises rapidly. By another, ion sources are placed inside the confinement zone and the field is changed in time, rapidly enough to prevent the reflected particles from hitting the sources. A further method has been used in the "DCX" (BARNETT *et al.* [1958]) and "OGRA" devices (ARTSIMOVICH [1958]), where ions of very high energy become trapped between the mirrors after having been created from molecular ions in a dissociation process. Part of these methods and similar arrangements can also be applied to devices of cusped geometry.

3.2. TOROIDAL FIELDS

Several plasma experiments are based on a study of high-current discharges in toroidal magnetic fields. The currents are generated in a torus-shaped chamber by means of electromagnetic induction effects. Investigations of such discharges have been undertaken in "ZETA" by THONEMANN *et al.* [1958] and BUTT *et al.* [1958] as well as in a great number of similar devices.

For a toroidal field the lines of force can be closed inside the confinement region and there is no escape of particles by the velocity $u_{\parallel}$. On the other hand, it was found in the example of § 2.1.(ii) that the boundary of the forbidden regions consists of a cylindrical surface. The latter runs through the

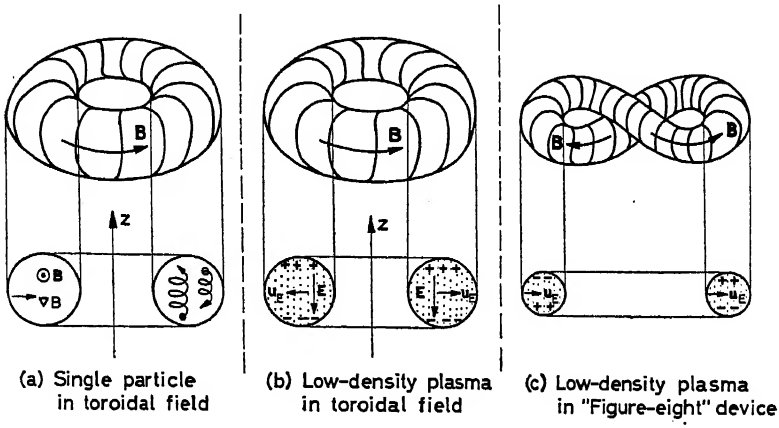


Fig. 7.8. Particle motions and associated separation of charge. (a) Single particle in toroidal field. (b) Plasma of low density in toroidal field (u_E is electric field drift). (c) Rotational transform in "Figure-eight" Stellarator gives a mean cancellation of electric field drift.

starting point of the particle and is parallel with the axis of symmetry. This result applies to a single particle moving in a vacuum field as sketched in Figure 7.8a. Equation (3.24) also shows that the magnetic gradient drift is directed along the z axis. In a torus of finite size single particles are therefore not contained by a toroidal field alone.

The situation is altered when an ionized gas is introduced into the toroidal field. Its ions and electrons will drift in opposite directions along the axis of symmetry according to equation (3.24). This produces a charge separation, and for a torus of finite size an electric potential will arise which varies along the axial direction. The analysis of § 2.1.(ii) then breaks down and we have to reconsider the situation in terms of the orbit theory. If the pressure of the gas is small compared to the magnetic energy density we can still assume the magnetic field to be a vacuum field which is produced by external sources only. The situation then becomes as demonstrated by Figure 7.8b. The axial charge separation by ∇B gives an axial electric field $\boldsymbol{E}$ which in its turn generates an electric field drift $\boldsymbol{u}_E = \boldsymbol{E} \times \boldsymbol{B}/B^2$. Consequently, the whole plasma escapes to the wall in the radial direction and there is no confinement in this case either.

To avoid these drawbacks and still use the advantage of field lines which are closed inside the confinement region Spitzer [1951] suggested the use of a *rotational transform* of the magnetic field in the "Stellarator" device. The simpliest way to achieve such a transform is to twist the original torus of Figures 7.8a and b into a "Figure-eight" device as shown by Figure 7.8c. The result of this is that the field lines do not close upon themselves after one turn around the configuration, as they do in an ordinary torus. Instead, a field line passing at a point P_1 of a cross section of the tube will pass the same section at points P_2, P_3, . . . after successive turns around the device (Figure

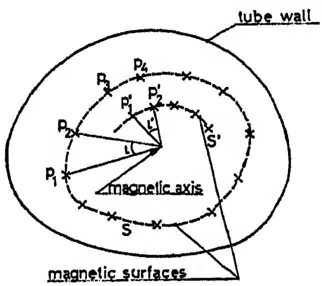


Fig. 7.9. Cross section of Stellarator tube. Points P_1, P_2, P_3, . . . represent successive intersections of a single magnetic field line (Spitzer [1958]).

7.9). A set of points P_1 will be transformed into a set of points P_2 after one turn; this is called a "magnetic transform" of the cross-sectional plane. We are especially interested in the case where the transform is primarily rotational in the sense that at least the outer portions of the plane all rotate in the same direction. Then, there must be at least one point in the plane which is transformed into itself. In the Stellarator systems there is one such point, i.e., the "magnetic axis" of Figure 7.9. KRUSKAL [1952] has shown that any point, other than that defining the magnetic axis, will not move far from a single closed curve when followed through successive magnetic transforms. A single line of force will therefore generate a "*magnetic surface*" after many circuits around the tube (Figure 7.9).

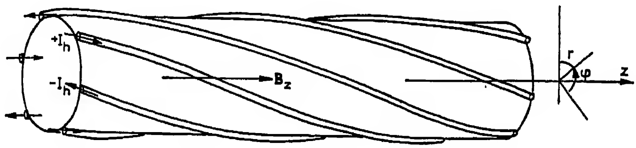


Fig. 7.10. Helical windings with currents $+I_h$ and $-I_h$ in alternate directions generate field components B_r and B_φ which produce a rotational transform together with the axial field B_z

For the "Figure-eight" device all points of a cross section will rotate the same angle ι after one turn around the tube. There is an alternative way in which a rotational transform can be produced, namely by the application of helical windings around the tube of an ordinary untwisted torus. This is demonstrated in Figure 7.10 for a part of the tube where the curvature has been omitted for the sake of simplicity. With currents $+I_h$ and $-I_h$ in alternating directions in these windings an additional field with components B_r and B_φ is superimposed on the field B_z along the tube axis. By this arrangement a rotational transform is generated which produces different angles of rotation, ι and ι', for points P_1, P_2 and P'_1, P'_2 at different magnetic surfaces S, S' (Figure 7.9).

For a detailed analysis of the present situation reference is made to the original papers by SPITZER [1958] and KRUSKAL *et al.* [1958]. Here we shall only make a brief physical consideration of the consequences of the rotational transform. First notice that the directions of the magnetic gradient drift and of the resulting electric field drift are such as to cancel for a particle during a complete transit through the tube of Figure 7.8c. There is a difficulty with this cancellation for particles which have too small or too

large values of $u_{\|}/W$, since these particles may get lost before a cycle has been completed. In the former case, the particles move too slowly along the field lines and will have time to drift to the walls during part of the cycle. The transverse drift is then produced by W^2 in combination with a transverse magnetic gradient as indicated by the first term of equation (3.24). In the latter case, the drift due to the curvature of the magnetic field lines dominates and forces the particles to the walls before a cycle has been completed. This drift is proportional to $u_{\|}^2$, whereas the longitudinal displacements are proportional only to $u_{\|}$. The present loss mechanisms can be minimized in the "Figure-eight" device by means of "scallops" with "corrugated" magnetic fields at the curved end sections.

In connexion with the rotational transform we also observe that space charges generated by the magnetic gradient drift can be short-circuited along the field lines, since the latter do not close upon themselves as in an ordinary torus.

With the rotational transform produced by helical windings the superimposed components B_r and B_φ form a field which resembles that of the cusped and "Picket Fence" geometry in Figure 7.7. The resulting Stellarator field has angles of rotation ι which differ for magnetic surfaces at different distances from the magnetic axis. The pitch of the field lines will therefore be non-uniform over the cross section of the tube, i.e., the field is "*sheared*". From energy considerations later discussed in Ch. 8, § 2.1 we should then expect the stability of the arrangement to become improved.

So far we have only considered the vacuum field of the Stellarator. When the pressure of a confined plasma cannot be neglected perturbations of the field and of the drift motions will be introduced by the plasma itself. This changes the rotational transform and may sometimes "unwind" it or produce an angle ι equal to a multiple of 2π. From this simple picture one would then expect the cancellation of space charges and stabilization by the rotational transform to vanish at a certain critical value of the axial plasma current. The existence of such a limit has also been proved rigorously by KRUSKAL [1954] and KRUSKAL *et al.* [1958].

Another difficulty with the rotational transform is that it has to short-circuit the separated charges in the longitudinal direction, i.e. along distances comparable to and exceeding the perimeter of the device. The presence of a finite electrical conductivity may therefore weaken the effects of the longitudinal short-circuit and of the rotational transform.

It is evident that a rotational transform will be produced by a plasma current which is flowing around a simple torus. The present discussions appear to indicate that this should have a suppressing action on the leakage by transverse drift motions and on instabilities. One might therefore ask if there is any essential difference at all between the Stellarator and an ordinary toroidal discharge. The answer is that a rotational transform with shear is already present in the vacuum field of the Stellarator. This makes possible a steady state confinement with nonzero pressures and axial currents. The confinement is insensitive to a large group of instabilities as found by JOHNSON *et al.* [1958].

3.3. ASYMMETRIC CONFIGURATIONS

In connexion with the helical field of the Stellarator we have already touched the problem of asymmetric perturbations of the confining field. The transverse drift motions arising from the corresponding magnetic field inhomogeneities are cancelled during a cycle around the device. A somewhat analogous situation arises in an arrangement recently described by IOFFE [1962] with the purpose to stabilize a magnetic mirror device. This is achieved if a number of conductors are introduced around the circumference of the mirror field in Figure 7.1a. The conductors should mainly follow the lines of the field and carry electric currents in alternating directions, somewhat like the windings of Figure 7.10. The resulting field will then become a superposition of the mirror field and a field similar to that of the "Picket

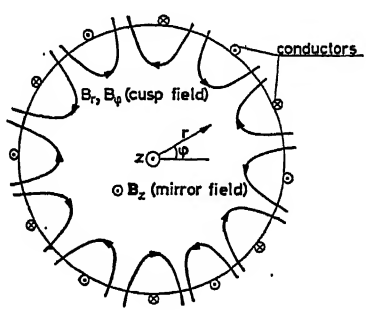


Fig. 7.11. Cross section along equatorial plane of mirror device with auxiliary windings which produce superimposed cusp field (cf. IOFFE [1962])

Fence" (Figure 7.7b). A cross section along the equatorial plane of the device would become as sketched in Figure 7.11. With a proper choice of

the parameters the field strength can be made to increase in the radial direction as in a cusped field, and the field lines will still run out through the mirrors before they reach a wall. In this device no cancellation of the drifts arising from the components B_r and B_φ can be obtained by a rotational transform along the field lines. However, the regularity of the arrangement still suggests that there is a rather well expressed cancellation of the drifts for particles which oscillate repeatedly up and down along the field lines and drift around the configuration in the φ direction at the same time.

A similar situation is faced also in connexion with particles travelling around the configuration of Figure 7.6 including the magnetically screened leads of Figure 7.2. The latter introduce a regular type of asymmetry and one should expect the displacements by drift motions to cancel at least in a first approximation. In a small region around the zero lines of Figure 7.2 the field is very weak and in the corresponding cylindrical case of Figure 7.6 there are some field lines leading from the confinement region out to external space. This does not imply, however, that particles necessarily get lost here. This is so, because the spatial extensions of the weak field region may become quite small compared to the Larmor radius within the same region. Particles approaching the latter from the interior of the plasma will all have velocity vectors situated close to the plane of Figure 7.2.

The transverse particle drifts in a toroidal magnetic field cannot only be cancelled by a rotational transform of the magnetic field lines. A rotational particle drift around the magnetic axis can equally well be produced by means of a "corrugated" magnetic field such as in a "bumpy" torus (cf. ARTSIMOVITCH [1958]). The strength of the toroidal magnetic field is then made to fluctuate periodically around the perimeter of the torus. The magnetic gradient drift arising from these inhomogeneities has the same effect on the transverse particle leakage as a rotational transform.

Even in configurations where the asymmetries are quite irregular, the confinement need not necessarily be spoiled. This has been shown by NORTHROP and TELLER [1960] and is also outlined in Ch. 4, § 1.5. The result depends on the constancy of the longitudinal invariant J which can be defined for fields like that given in Figure 4.6. Here the particles oscillate between two mirror points and the field strength has only one minimum between the points on a field line.

* 4. Mechanisms for Particle Losses

Up to this stage we have treated the confinement of a collisionless plasma,

partly on the basis of the first order orbit theory. This paragraph summarizes a number of non-adiabatic effects and dissipative mechanisms which may produce particle losses in a real plasma.

4.1. NON-ADIABATIC EFFECTS

Due to the finite Larmor radius the results of the perturbation theory of Chapters 3 and 4 are only approximately valid and the adiabatic invariants are no exact constants of the motion. The examples of Figures 4.3 and 4.9 show that the deviations from constancy can become quite large when conditions (3.1) and (3.2) no longer hold. In a study of the boundaries of the forbidden regions we can use the results (7.7) and (7.36) for symmetric configurations, also when the basic assumptions of the perturbation theory become invalid. The end losses of a mirror device are more difficult to calculate explicitly for particles not encircling the axis of symmetry. In any case recent investigations by GARDNER [1962] show that particles outside the loss cones of a strong magnetic mirror field are contained for all times.

To determine the particle losses due to deviations from adiabatic invariance there still exists the possibility of using numerical computers. This has been done for a mirror device by GARREN *et al.* [1958] who arrive at the following results:

(i) The variations of the equivalent magnetic moment M during one cycle of motion between the magnetic mirrors were found to be quite large for Larmor radii of practical interest.

(ii) The fluctuations of M are nearly periodic and the residual change in M between successive reflections appears to be rather small. It was also found that the cumulative effects of many traversals were quite small for orbits which intersect the equatorial plane of the device in points which could be joined by smooth curves. Other orbits were found where this was not the case and the particles escaped.

(iii) The relative fluctuations in M were found to decrease as exp (− const. $\omega_g s_m/w$), where ω_g is the gyro frequency, $2s_m$ the distance between the mirrors and w the particle velocity. This is what should be expected from the theoretical results by HERTWECK and SCHLÜTER [1957] and by KRUSKAL [1958] (see also Chapter 4, § 2.3).

(iv) The effect of magnetic-field imperfections was studied by introducing

asymmetries in the field. In some cases it was then found that particles were lost through the mirrors.

Non-adiabatic losses have further been considered by GRAD and VAN NORTON [1962] for a cusped magnetic field. They find that there are three types of orbits, i.e. one class of highly adiabatic orbits which are contained indefinitely, one of highly non-adiabatic orbits which undergo substantial changes, and an intermediate class where the changes are relatively small and uncorrelated.

A theoretical investigation of the losses from a magnetic trap is also due to CHIRIKOV [1960]. He finds that particles escape according to a resonance phenomenon which occurs between the longitudinal oscillations and the Larmor motion for particles which are trapped between two magnetic mirrors. These results are in agreement with experiments made by RODIONOV [1960].

Recent investigations on the containment of charged particles in an asymmetric mirror geometry have been reported by GIBSON *et al.* [1962]. Two mirror coils were tilted from their symmetric positions to form a section of a "bumpy" torus. Experiments on the trapping of positrons emitted from a radioactive source into this field indicated that the latter were contained for many seconds.

4.2. DIFFUSION IN COORDINATE SPACE

When non-identical, charged particles collide their guiding centra will be displaced and this produces a transverse diffusion across the magnetic field. Consider a collision between two particles of masses m_1 and m_2, charges q_1 and q_2 and velocities $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$. Their radii of gyration are given by the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ directed from the corresponding guiding centra to the positions of the particles. After the collision corresponding quantities will be denoted by a prime. The vector connecting the centre of mass, CM, of the guiding centra and the point of collision is initially

$$\begin{aligned}\boldsymbol{a}_{\mathrm{CM}} &= (m_1\boldsymbol{a}_1 + m_2\boldsymbol{a}_2)/(m_1 + m_2) \\ &= \boldsymbol{B}_0 \times \left(\frac{m_1^2}{q_1}\boldsymbol{w}_1 + \frac{m_2^2}{q_2}\boldsymbol{w}_2\right)/(m_1 + m_2)B_0^2,\end{aligned} \tag{7.43}$$

where the magnetic field is assumed to be homogeneous and the drift velocity $\boldsymbol{u}$ is neglected compared to the velocity $\boldsymbol{W}$ of gyration. Conservation of momentum requires

$$m_1\boldsymbol{w}_1 + m_2\boldsymbol{w}_2 = m_1\boldsymbol{w}_1' + m_2\boldsymbol{w}_2' \tag{7.44}$$

and the displacement of CM therefore becomes

$$\boldsymbol{a}'_{\mathrm{CM}} - \boldsymbol{a}_{\mathrm{CM}} = m_1 \boldsymbol{B}_0 \times (\boldsymbol{w}'_1 - \boldsymbol{w}_1) \cdot \left(\frac{m_1}{q_1} - \frac{m_2}{q_2} \right) / (m_1 + m_2) B_0^2. \tag{7.45}$$

Consequently, CM will be displaced only for particles with different values of m/q.

Next consider an assembly of identical particles of mass m, charge q and constant density n in an inhomogeneous magnetic field $\boldsymbol{B}$. In the first approximation the position of the centre of mass of the guiding centra is obtained from a summation over all particles of a volume element (cf. SPITZER [1956]):

$$\langle \boldsymbol{C} \rangle = \frac{1}{n} \sum_{\nu} (\boldsymbol{\rho}_\nu - \boldsymbol{a}_\nu) = \frac{1}{n} \sum_{\nu} (\boldsymbol{\rho}_\nu + m \boldsymbol{W}_\nu \times \boldsymbol{B}/qB^2), \tag{7.46}$$

where $\boldsymbol{W}_\nu$ is the velocity of gyration. The velocity of $\langle \boldsymbol{C} \rangle$ becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \boldsymbol{C} \rangle = \frac{1}{n} \sum_{\nu} \left[\boldsymbol{w}_\nu + m \frac{\mathrm{d}\boldsymbol{W}_\nu}{\mathrm{d}t} \times \boldsymbol{B}/qB^2 + m \boldsymbol{W}_\nu \times \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{B}/qB^2) \right]. \tag{7.47}$$

In absence of electric fields, and in presence of a force $\boldsymbol{f}_{\mu\nu}$ due to collisions between the μ-th and ν-th particles, the equation of motion of the ν-th particle would become

$$m \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{u}_\nu + \boldsymbol{W}_\nu) = q \boldsymbol{w}_\nu \times \boldsymbol{B} + \sum_{\mu} \boldsymbol{f}_{\mu\nu}. \tag{7.48}$$

After insertion of $\mathrm{d}\boldsymbol{W}_\nu/\mathrm{d}t$ from (7.48) into (7.47) the result becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \boldsymbol{C} \rangle = \frac{m}{n} \sum_{\nu} \left[\boldsymbol{W}_\nu \times \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{B}/qB^2) - \frac{\mathrm{d}\boldsymbol{u}_\nu}{\mathrm{d}t} \times \boldsymbol{B}/qB^2 + \boldsymbol{w}_{\nu\parallel}/m \right]$$

$$+ \frac{1}{n} \sum_{\mu} \sum_{\nu} (\boldsymbol{f}_{\mu\nu} \times \boldsymbol{B}/qB^2). \tag{7.49}$$

The total collisional force $\Sigma_\mu \, \Sigma_\nu \, \boldsymbol{f}_{\mu\nu}$ should vanish according to Newton's third law since the collisions produce an internal force on the system. In case of a homogeneous magnetic field we therefore have $(\mathrm{d}\langle \boldsymbol{C} \rangle/\mathrm{d}t)_\perp = 0$, in agreement with (7.45) for $m_1/q_1 = m_2/q_2$. For an inhomogeneous field the two first terms inside the square bracket of (7.49) will give a contribution which is due to the magnetic gradient drift of (3.24). At the same time the last term of equation (7.49) does no longer vanish. It will be of the order of the ratio d_c/L_{cB} of the mean distance d_c between particles during collisions

and the characteristic length L_{cB} of the magnetic field variation. Thus, there should arise a displacement of the centre of mass by the combined action of collisions and a magnetic gradient.

From the simple considerations of (7.45) we have seen that the centre of mass of the guiding centra does not move in the transverse direction under the influence of like-particle collisions in a homogeneous magnetic field. Care, however, necessary when this result is used to conclude that identical particles do not produce any transverse diffusion across such a field. The drift of the centre of mass of the guiding centra is not necessarily equal to the diffusion velocity of the plasma. A further analysis by means of the Boltzmann equation can be applied to the flux of particles in the diffusion process. This shows in fact that the first order flux vanishes for collisions between identical particles (LONGMIRE and ROSENBLUTH [1956]).

In higher orders there will also exist a slow diffusion for like-particle collisions due to higher derivatives of the density gradient (SIMON [1955]).

The transverse diffusion rate for non-identical particles is cut down considerably in a strong magnetic field as suggested by (7.45) and which also becomes obvious from a macroscopic diffusion theory. As long as there are no instabilities in the plasma, of large or small scale, the transverse particle losses should therefore become quite small in such a field.

The presence of neutral particles produces additional losses by encounters. Elastic collisions between charged and neutral particle enhances the transverse diffusion rate of the plasma. Further, the neutrals may exchange charges with the ions. Such collisions have the same effect as a loss of energetic plasma particles. Processes of this kind have been investigated in the DCX device by BARNETT *et al.* [1958].

4.3. DIFFUSION IN VELOCITY SPACE

Particles are lost not only by diffusion in coordinate space but also by diffusion in velocity space, e.g., into the loss cones of a mirror machine. Coulomb collisions cannot be treated strictly as encounters between rigid spheres, because their potential of interaction has a long range. Instead the diffusion process is mainly determined by cumulative effects from distant encounters. For the diffusion into the loss cones we further observe that collisions between identical particles produce first order effects. Thus, there is no analogy with the cancellation of the diffusion rate for like-particle collisions in coordinate space.

A detailed treatment of this problem is due to CHANDRASEKHAR [1942]

and SPITZER [1956]. Only part of the main results will be presented here. Assume a "test particle" of mass m, charge Ze and velocity $\boldsymbol{w}$ which collides with "field particles", all of mass m_1 and charge $Z_1 e$.

For close encounters we need only consider the collision between a pair of particles. The distance of closest approach, i.e. the "impact parameter", becomes $e^2 ZZ_1/4\pi\varepsilon_0 mw^2$ for an angular deflection of 90°. We can define a sphere with this distance as radius, as well as a corresponding mean free path. When there is a density n_1 of the field particles the time

$$t_{cl} = 8\pi\varepsilon_0^2 m^2 w^3/e^4 n_1 Z^2 Z_1^2 \tag{7.50}$$

becomes an approximate measure of the collision time for close encounters.

For distant encounters the long range of the Coulomb forces has to be taken into account. Thus, we have to calculate the effect of many small deflections which occur simultaneously. A time t_D for a gradual deflection which reaches 90° can be defined by

$$t_D = w^2/\langle(\Delta w_t)^2\rangle, \tag{7.51}$$

where Δw_t is the transverse deflection of $\boldsymbol{w}$ with respect to its initial direction $\hat{\boldsymbol{w}}$ and the mean value given by the denominator indicates the increase of velocity dispersion per second due to encounters. With $l_1 = (m_1/2kT_1)^{\frac{1}{2}}$ and T_1 being the temperature of the gas consisting of the field particles we have

$$t_D = 2\pi\varepsilon_0^2 m^2 w^3/e^4 n_1 Z^2 Z_1^2 \ln \Lambda_D \cdot \Psi(l_1 w), \tag{7.52}$$

where

$$\Lambda_D = (12\pi/e^3 ZZ_1) \cdot (k^3 T^3 \varepsilon_0^3/n_e)^{\frac{1}{2}}, \tag{7.53}$$

n_e is the electron density and

$$\Psi(l_1 w) = \psi_D - \frac{[\psi_D - l_1 w \, d\psi_D/d(l_1 w)]}{2(l_1 w)^2}, \tag{7.54}$$

$$\psi_D(l_1 w) = (2/\pi^{\frac{1}{2}}) \int_0^{l_1 w} e^{-y^2} \, dy. \tag{7.55}$$

Both $\ln \Lambda_D$ and $\Psi(l_1 w)$ are rather slow functions of their arguments; the former is of the order 20 and the latter of the order unity in many cases of practical interest.

The results (7.52) – (7.55) have been deduced under the assumption that the maximum impact parameter can be put equal to the Debye distance, $(\varepsilon_0 kT_e/e^2 n_e)^{\frac{1}{2}}$. In a more refined theory ROSTOKER and ROSENBLUTH [1960]

have evaluated the frictional drag on the test particle without such a limitation on the impact parameter. As well as PINES and BOHM [1952] they also include the drag due to plasma-wave emission.

The present results can be used in a calculation of the rate of diffusion into the loss cones. When the mirror ratio and the angle θ_m are small the probability for a particle to fall into the loss cones of Figure 7.4 by large angle scattering is simply

$$P_{cl} = 1 - \cos\theta_m \approx \tfrac{1}{2}\sin^2\theta_m = \tfrac{1}{2}B_0/B_m \tag{7.56}$$

as given by (7.40) in absence of an electric field. For distant encounters the corresponding probability becomes

$$P_D \approx \frac{1}{1 + \ln(B_m/B_0)} \tag{7.57}$$

according to JUDD *et al.* [1955]. Observe that the probability P_D only has a slow, logarithmic dependence on the mirror ratio. This is due to the cumulative effects of distant encounters.

The plasma decay by diffusion into the loss cones proceeds at a rate (cf. LINHART [1960]):

$$\frac{dn}{dt} = -n_1\left(\frac{P_{cl}}{t_{cl}} + \frac{P_D}{t_D}\right). \tag{7.58}$$

If expressions (7.50) – (7.57) are inserted and integration is carried out, the result becomes

$$n(t) = n_0\,\tau_w/(t + \tau_w), \tag{7.59}$$

where

$$\tau_w = \frac{8\pi\,\varepsilon_0^2 n_0 m^2 w^3}{e^4 n_1^2 Z^2 Z_1^2}\cdot\left[\tfrac{1}{2}\frac{B_0}{B_m} + \frac{4\ln\Lambda\ \Psi(l_1 w)}{1 + \ln(B_m/B_0)}\right]^{-1} \tag{7.60}$$

is the corresponding decay (or scattering) time and n_0 is the initial density. For values of practical interest the second term within the square bracket is usually of the order 100 which is much larger than the first term within the same bracket. The scattering time τ_w is therefore mainly determined by the distant encounters and is increased only very slowly by an increase in the mirror ratio, B_m/B_0. Unfortunately, this limits the effectiveness by which the end losses along the field lines can be reduced in a mirror machine.

As an example we may mention that a deuterium plasma of density $n = 10^{20}$ m^{-3} and temperature $T = 10^5$ °K will escape through the loss cones in a time of the order of $\tau_w = 10^{-7}$ seconds. If the temperature is

raised to $T = 10^9$ °K, the corresponding time becomes about 0.5 seconds. In this latter case the power produced by thermonuclear reactions in a tritium-deuterium plasma would become about four times larger than the power loss due to diffusion into the loss cones.

4.4. AMBIPOLAR EFFECTS

The diffusion rates of ions and electrons are in general different both in coordinate space and in velocity space. This produces space charges and an ambipolar electric field which in its turn affects the escape rate of the particles. A detailed treatment of the effect is due to KAUFMAN [1956].

Especially if the electron temperature T_e is much smaller than that of the ions, T_i, so that $m_2^e w_3^e \ll m_i^2 w_i^3$ the scattering time (7.60) of electrons into the loss cones of a mirror device becomes shorter than that of the ions. The central parts of the plasma will then become positively charged. The mirror ratio R_m for particles at the boundary of the loss cones is given by equation (7.40). In the present situation we will have $\phi_0 - \phi_m > 0$ and of the order of kT_e/e. Further $m_i w_0^2 = 2kT_i$ for ions and $\sin^2 \theta$ is of order unity. The effective mirror ratio for ions in a corresponding device with $\phi_0 = \phi_m$ and with the same value of θ then becomes

$$R_{\text{eff}} \approx \frac{R_m}{1 + c_1 T_e/T_i}, \tag{7.61}$$

where c_1 is a constant of order unity. The effective mirror ratio is thus reduced and this shows that the ion losses are enhanced by the ambipolar field which tends to pull the ions out through the mirrors.

4.5. INSTABILITIES

In addition to the drift motions in external fields and to the loss processes discussed so far there may arise cooperative phenomena inside the plasma itself, by which the particles are driven out of the confinement region. Consequently, deviations from the equilibrium state sometimes become unstable and the charged particles interact with themselves and with the external fields in a way to spoil the confinement. The instabilities can be of a more or less irregular character and occur both in coordinate space and in velocity space.

Reviews of these phenomena have been given by several authors. In this book we shall only treat a certain type of instability in Chapter 8, mainly from the point of view of the single particle picture.

CHAPTER 8

STABILITY

The purpose of the present chapter is not to give a complete survey of plasma stability. It merely presents some illustrations to the subject with special attention to the motion of individual particles.

In general there exist two classes of instability phenomena, denoted as magnetohydrodynamic instabilities (or macroinstabilities) and micro-instabilities. In this chapter we shall only concentrate on a restricted part of the former, namely on the so called "flute type" instability. We assume the plasma particles to have Maxwellian velocity distributions and use macroscopic fluid equations to describe their motion in space. Thus, we shall not present any treatment of the motions in velocity space and their connexion with microinstabilities. The discussion will further be limited to small plasma energies corresponding to that of the magnetic field, and electromagnetic induction effects are excluded which give rise to instabilities such as those of the "kink" and "sausage" types.

Non-linear stability theory is still in an early stage of development. The theory to be presented here is linear and does not take into account what happens when the amplitude of an unstable disturbance grows to large values. This fact should be kept in mind since it restricts severely the application of linear theory to actual physical problems.

1. Principles

The stability of a system, originally at equilibrium, can in principle be treated in two ways:

(i) A disturbance is resolved into elementary modes and the development of the latter is studied as a function of time.

(ii) A virtual displacement is performed on the system from its equilibrium state and the change in the potential energy is deduced.

The first method, which often is denoted as the "principle of normal modes" has the merit of its relative simplicity and that it takes dynamical effects into direct account. It is especially suited for linearized problems

where the growth of small perturbations of the equilibrium state are to be studied. Instability of the system manifests itself as an unlimited growth of the perturbation amplitudes. A drawback of this method is that it is not well suited for a treatment of non-linear phenomena, i.e. for perturbations of large amplitudes. Conclusions about stability from a linearized theory should be taken with care. There may very well exist systems with positive growth rates for small disturbances, but which are limited by non-linear effects such as to make the system stable from the practical point of view. A reverse situation is also imaginable.

The second method, which is often called the "energy principle", has earlier been applied to physical problems in general and was first introduced into magnetohydrodynamics by LUNDQUIST [1951] and further developed by BERNSTEIN *et al.* [1958]. It makes stability considerations possible even for systems of rather complicated geometry, and is not necessarily restricted to small perturbations. The advantage of this method lies in the fact that, if only stability is to be determined and not the growth rates or oscillation frequencies of particular types of disturbances, it is only necessary to discover if there is any perturbation which decreases the potential energy or not. One may certainly state that the system must be stable if the equilibrium has a lower potential energy than all neighbouring states. On the other hand, if there exists a neighbouring state with lower potential energy than the equilibrium it is not always quite clear that the former can be reached immediately. The transition between the states and the rate at which it takes place are determined by the equations of motion. As a matter of fact, the latter are also included in a rigorous analysis based upon the energy principle.

Finally, a theoretical development of the energy principle has to be modified when energy is being fed into the system by external sources.

2. Flute Disturbances

In this paragraph we shall study a plasma of *low* energy density confined in a *strong* magnetic field. The latter is then nearly a vacuum field. Any distortion of the confining field increases its energy by amounts far greater than those available in the plasma. At a first sight, one may conclude from this that the plasma should be effectively trapped in the field, provided that the latter constitutes a suitable magnetic bottle in the equilibrium state. However, the plasma may itself set up electric fields and additional drift motions by which it is able to escape across the field, leaving the latter unchanged at

the same time. A phenomenon of this kind is the so called "flute" instability which will be the subject of the subsequent paragraphs.

Starting with the one-fluid equations of magnetohydrodynamics KRUSKAL and SCHWARZSCHILD [1954] proved the instability of a plasma which is supported under gravity by a magnetic field. Their result is closely related to the Rayleigh-Taylor instability in hydrodynamics. A further discussion on plasma stability was inspired by TELLER [1954] who suggested that the "interchange" between parts of a plasma and its confining magnetic field may lead to instabilities for certain configurations. ROSENBLUTH and LONGMIRE [1957] were the first to make a closer examination of the situation in terms of particle orbits. Almost simultaneously the problem was also analysed by BERKOWITZ *et al.* [1958] in terms of a macroscopic fluid model.

A study of the sources of the "flute" phenomenon and its development will now be presented.

2.1. PHYSICAL DISCUSSION OF INSTABILITY MECHANISM

Consider a plasma of uniform and low density n supported against a homogeneous gravitation field $\boldsymbol{g}$ by a homogeneous and strong magnetic field $\boldsymbol{B}$, as shown in Figure 8.1. In the equilibrium state the plasma is supposed to have a sharp boundary along the surface $y = 0$, perpendicular to $\boldsymbol{g}$, and the lower half plane is a vacuum region. The magnetic field which is induced by the plasma currents should be small compared to the externally imposed field $\boldsymbol{B}$.

Now make a displacement of the boundary so that it obtains the sinusoidal shape given by Figure 8.1 in all planes perpendicular to $\boldsymbol{B}$. The

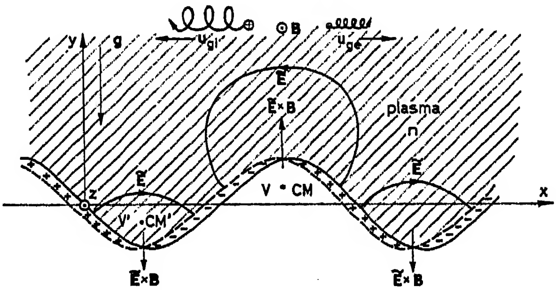


Fig. 8.1. Plasma supported against the gravitation force $\boldsymbol{g}$ by a magnetic field $\boldsymbol{B}$. Charge separation occurs at a sharp boundary which divides the plasma from vacuum and electric field drifts are generated.

resulting change of state can be considered as a displacement of fluid from the region of volume V to the region of volume $V' = V$ in the figure. These two regions have cross-sectional areas of a sinusoidal half-wave and extend in the z direction. Since they are equal there is no change in the magnetic flux enclosed by the plasma. By the interchange of fluid from V to V' the corresponding centre of mass CM in V is lowered to the position CM′ in V'. The same thing happens for all pairs V, V' of elements along the boundary and the potential energy is thus decreased by the displacement.

If instead gravity would act in the opposite direction the potential energy would increase. The latter situation is true for all possible perturbations and the system should therefore be stable in such a case.

When gravity acts downwards as in Figure 8.1 the potential energy has just been found to decrease, and one would therefore guess that the special mode pictured here should become unstable. However, the present simple arguments form no rigorous proof, since it has also to be shown that the plasma particles are able to move in such a way that states of decreasing potential energy actually can be reached. This will be the case for the mode in Figure 8.1, but not always for that in Figure 8.5a.

For further information about this question we therefore investigate the particle motions at the boundary. The external force drift of equation (3.23) becomes $\boldsymbol{u}_{gi} = m_i\boldsymbol{g} \times \boldsymbol{B}/eB^2$ for ions and $\boldsymbol{u}_{ge} = -m_e\boldsymbol{g} \times \boldsymbol{B}/eB^2$ for electrons. These velocities are oppositely directed as in Figure 8.1 and produce a charge separation at the plasma boundary, as soon as it becomes rippled. The electric field $\tilde{\boldsymbol{E}}$ which arises from the charges produces a drift $\tilde{\boldsymbol{E}} \times \boldsymbol{B}/B^2$ which enhances the crests and troughs of the perturbed boundary. The perturbation therefore grows and this makes the ripples steeper and the charge separation by the horizontal drifts $\boldsymbol{u}_{gi}$ and $\boldsymbol{u}_{ge}$ increases. Thus, we obtained the required mechanism which drives the plasma across the magnetic field at the vacuum boundary, leaving the field uninfluenced at the same time. As a result, the plasma will in the present case be able to reach states of lower energy and becomes unstable. However, we shall also see from a more detailed analysis in the coming sections that there are effects which can stabilize the flute disturbances under certain conditions, even if there seem to exist neighbouring states of lower energy.

According to the orbit theory a gravitation field is not the only means by which charge separation can occur at a plasma boundary. Inertia forces of an accelerated plasma body may serve the same purpose. Of greatest interest

in connexion with experiments is perhaps the gradient drift which occurs from the inhomogeneity of the confining magnetic field itself.

As an example on the effect of a magnetic gradient, we may consider the cylindrical shell of plasma confined in the field from an axial line current as demonstrated in Figure 8.2. Suppose that the outer boundary is deformed by a wave-like perturbation such as to produce no change to lowest order in the enclosed magnetic flux; this is a necessary condition for the creation of a flute disturbance which has to leave the magnetic field nearly undistorted. The magnetic fluxes contained in the volume elements V and V' of Figure 8.2

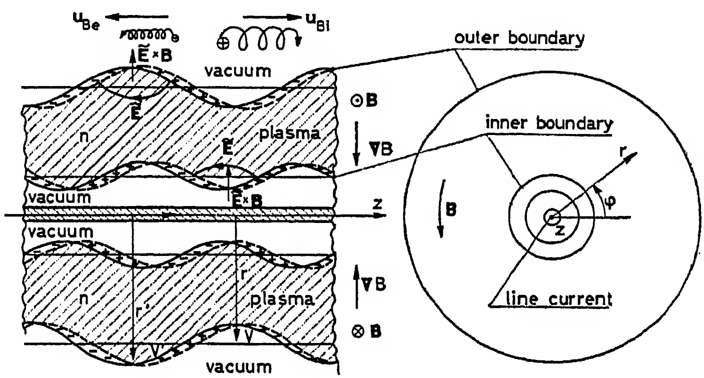


Fig. 8.2. Cylindrical shell of plasma confined in the magnetic field from an axial line current.

should therefore be approximately equal. Thus, $\langle B \rangle \cdot S = \langle B' \rangle \cdot S' = \Phi$, where $\langle B \rangle$ and $\langle B' \rangle$ are the mean magnetic field strengths inside the elements and S and S' are the areas of the corresponding cross sections. The change in volume of the plasma when matter disappears from V and goes into V' is therefore given by the change from $2\pi r/\langle B \rangle$ to $2\pi r'/\langle B' \rangle$. For the outer boundary of Figure 8.2 the quantity r/B increases with r and the volume V' is therefore larger than V. This implies that the plasma can expand without changing the magnetic field noticeably, and the perturbation should therefore give a state of lower potential energy. For the inner boundary a wave-like perturbation has the opposite result. We therefore conclude that the inner boundary of the shell is stable and expect the outer to become unstable. The latter is then subject to an "interchange" instability where matter leaves the volume V and enters V'.

After an interchange of the volume elements there should arise a corre-

sponding change in the pressure balance and in the magnetic field at V and V'. However, as long as the pressure is much smaller than the magnetic energy density, the change in the magnetic field will affect the enclosed flux Φ and the areas S and S' only very little. The present conclusions about the sign of the changes in potential energy are therefore not influenced by the detailed pressure balance.

For a magnetic field geometry of more complicated shape, like the mirror geometry of Figure 8.3, we can write the condition for stability as

$$\delta V = V' - V = \Phi\delta \int \frac{\mathrm{d}s}{B} < 0, \tag{8.1}$$

where integration should take place along a field line at the boundary, the flux Φ enclosed by the volumes V and V' should be constant, and s is the coordinate along $\boldsymbol{B}$. Since curl $\boldsymbol{B} \approx 0$ the line integral around the small shaded area of Figure 8.3 should vanish, and this yields $\delta R/R = \delta B/B$, where R is the radius of curvature of a field line. Observe also that the magnetic flux $\delta\Phi = 2\pi r B\, \delta R$ between the two lines L_0 and L_0' is constant in the direction along the latter. With these relations inserted condition (8.1) becomes

$$\int \frac{\mathrm{d}s}{rB^2R} > 0 \tag{8.2}$$

as shown by ROSENBLUTH and LONGMIRE [1957]. Note that R is positive at the ends of the configuration of Figure 8.3 and negative in the middle region.

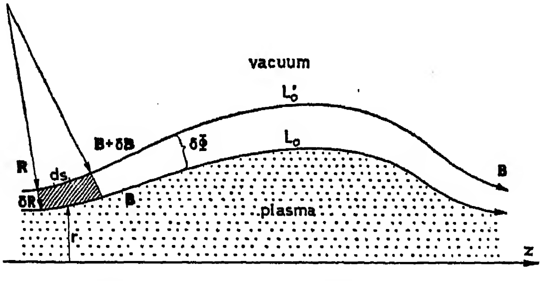


Fig. 8.3. Relations between strength and radius of curvature of the magnetic field.

The contributions from the latter will dominate and tend to make the system unstable.

When a sheared field is present as described in Chapter 7, § 3.2 we observe

that the interchange of fluid elements can only take place if there arises a distortion of the latter, and this improves stability.

An examination of the magnetic gradient drifts due to the inhomogeneity and curvature of the magnetic field shows that a charge separation occurs at the boundaries of the shell in Figure 8.2. The resulting electric field has such a direction that the perturbation amplitude is increasing at the outer boundary but is decreasing at the inner one. Consequently, we have here again the required mechanism which causes the assumed displacements of the outer boundary to grow, but suppresses those of the inner, stable boundary.

The situation can be considered in the same way for a plasma-vacuum boundary in any type of inhomogeneous field, as indicated in Figure 8.4. The corresponding conclusions are that the boundary is stable if the magnetic field bends convexly towards the plasma, i.e. for the "cusped geometry" of Figure 8.4a. It may become unstable when the opposite is true, i.e., for the "mirror geometry" of Figure 8.4b. In the forthcoming discussion on

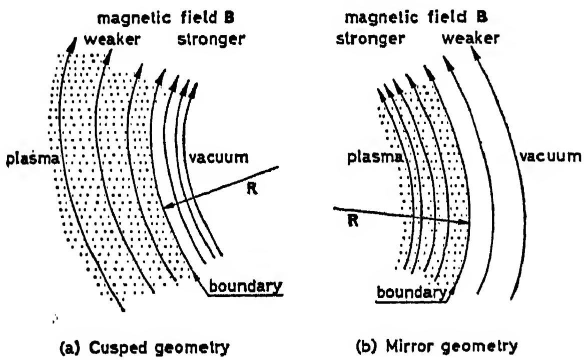


Fig. 8.4. Plasma confined in magnetic field of cusped geometry (a) and mirror geometry (b).

particle motions we will see that there are effects which may stabilize the flute disturbances under certain conditions, also in the case of mirror geometry.

We shall now start a more detailed investigation of the flute instability and the effects which influence its growth rate, as found in contributions by ROSENBLUTH and LONGMIRE [1957], LEHNERT [1961, 1962a, c], ROSENBLUTH *et al.* [1962] and ROBERTS and TAYLOR [1962].

The study which follows does not make any pretention to give an ex-

haustive analysis of all instabilities which may arise. Its purpose is merely to demonstrate how the flute mechanism operates and to draw attention to a number of effects which influence the corresponding growth rates. We shall investigate normal modes and make a simple *localized perturbation analysis* of disturbances in a limited region of the plasma. The latter is assumed to extend far away from the same region in all directions before it reaches its boundaries. The corresponding dispersion relations are then independent of the boundary conditions. Thus, we shall not solve the complete eigen-value problem in terms of the exact solution of the equilibrium state. In the analysis of the disturbed state we shall instead treat all undisturbed quantities as constants and afterwards examine the validity of the dispersion relations obtained from such a procedure. It should of course be kept in mind that such a localized perturbation analysis does not permit a general study of disturbances the scale of which is comparable to the macroscopic dimensions of the plasma and its boundaries.

2.2. BASIC RELATIONS

The starting points of the present treatment are the equations (5.17) and (5.20) expressing conservation of matter and momentum. The latter is equivalent to equations (3.46) and (3.47) deduced from the orbit theory in first order. At an early stage of our treatment we shall also rearrange the basic equations in a way to demonstrate the single particle behaviour of the plasma. Such a representation has already been touched in Ch. 5, § 2.3 in connexion with the drift of density distributions.

Consider the equation of motion (5.20). Since we shall also be interested in the stability of a rotating plasma we extend our basic relations such as to include a frame of reference which rotates at the angular velocity $\boldsymbol{\Omega}$. For simplicity assume the plasma to perform a stationary, rigid body rotation with constant $\boldsymbol{\Omega}$. According to the transformation (7.23) of Ch. 7, § 2.2 we now write the macroscopic equation of motion as

$$nm\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = nq(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}) + nm\boldsymbol{g} - \operatorname{div}\pi + \tfrac{1}{2}nm\nabla(\boldsymbol{\Omega}\times\boldsymbol{\rho})^2 + 2nm\boldsymbol{v}\times\boldsymbol{\Omega}, \quad (8.3)$$

where $\boldsymbol{g}$ is the acceleration due to gravity and all quantities refer to the rotating system. The quantity $-\tfrac{1}{2}(\boldsymbol{\Omega}\times\boldsymbol{\rho})^2$ represents the centrifugal potential. The origin of the coordinate system has been chosen on the axis of rotation.

We now denote ions and electrons by subscript $\nu =$ i, e and introduce

$d_\nu/dt = \partial/\partial t + \boldsymbol{v}_\nu \cdot \nabla$. Equation (8.3) becomes after vector multiplication by $\boldsymbol{B}$:

$$n_\nu \boldsymbol{v}_{\nu\perp} = n_\nu \boldsymbol{E} \times \boldsymbol{B}/B^2 + n_\nu m_\nu \boldsymbol{g} \times \boldsymbol{B}/q_\nu B^2 + \boldsymbol{B} \times \operatorname{div} \boldsymbol{\pi}_\nu/q_\nu B^2$$
$$- n_\nu m_\nu \frac{d_\nu \boldsymbol{v}_\nu}{dt} \times \boldsymbol{B}/q_\nu B^2 - 2n_\nu m_\nu \boldsymbol{B} \times (\boldsymbol{v}_\nu \times \boldsymbol{\Omega})/q_\nu B^2$$
$$+ \tfrac{1}{2} n_\nu m_\nu \nabla(\boldsymbol{\Omega} \times \boldsymbol{\rho})^2 \times \boldsymbol{B}/q_\nu B^2. \tag{8.4}$$

The present analysis will be based on the following assumptions and starting points:

(i) The plasma pressure is small compared to the magnetic energy density. Any transverse plasma motion which would produce noticeable distortions of the magnetic field is then energetically impossible, because it would increase the field energy far beyond the energy content of the plasma. Thus, the magnetic field is close to a vacuum field. We therefore put curl $\boldsymbol{B} = 0$ and consider only the flute-type of disturbances by which matter moves across the field, leaving the latter unchanged at the same time. The magnetic field from the plasma currents has then a negligible influence on the particle drifts. The external sources of the magnetic field are stationary.

(ii) We assume that the electric field in the plasma can be derived from an electric potential only, i.e. $\boldsymbol{E} = -\nabla\phi$. This implies that we decouple the phenomena to be studied from transverse modes such as electrodynamic and magnetohydrodynamic wave phenomena (cf. ROSENBLUTH *et al.* [1962]). The latter give rise to electromagnetic induction effects which increase the field energy. In other words, the disturbances which are to be studied should change slowly in time compared to the velocity of light and to the Alfvén velocity. The assumed form of the electric field is then easily verified by simple estimations of (8.4), (2.10) and (2.2). Since the external sources of the magnetic field are stationary and the induced field is neglected we can put $\partial \boldsymbol{B}/\partial t = 0$.

(iii) Since $m_e \ll m_i$ we neglect the inertia forces involved in the macroscopic motion of electrons compared to those acting on ions. We also neglect the dissipation produced by a finite resistivity. For further discussions on the effects due to finite current-carrier mass and to finite resistivity reference is made to recent investigations by FURTH [1962] and FURTH *et al.* [1962].

(iv) In the unperturbed state the centre of mass of the plasma is assumed to be at rest with respect to the frame of reference defined at the beginning of this paragraph. The plasma should then be in an electrically neutral state with density N and pressure tensors π_{i0} and π_{e0} of ions and electrons. The density gradient ∇N, the divergence div π_{i0} and div π_{e0} of the pressure tensors, the gravitation field $\boldsymbol{g}$ and the centrifugal force $\frac{1}{2}m\nabla(\boldsymbol{\Omega} \times \rho)^2$ are all assumed to be perpendicular to the magnetic field in the special applications to the theory. Since $m_e \ll m_i$ we put $\boldsymbol{v}_{i0} = 0$ in the initial state. A small electron current generates the force $- eN\boldsymbol{v}_{e0} \times \boldsymbol{B}$ which then supports the dilute plasma against the pressure, gravitation and centrifugal forces. Ions and electrons are coupled by the unperturbed electric field $\boldsymbol{E}_0$. With $\boldsymbol{v}_{i0} = 0$ and curl $\boldsymbol{B} \approx 0$ the unperturbed ion pressure tensor π_{i0} is assumed to be given in terms of ∇B and the unperturbed scalar pressures $P_{\parallel i}$ and $P_{\perp i}$ as expressed by equation (5.24). Then,

$$\boldsymbol{E}_0 \approx - (m_i/e)\boldsymbol{g} + (1/eN)(\operatorname{div} \pi_{i0})_\perp - \tfrac{1}{2}(m_i/e)\nabla(\boldsymbol{\Omega} \times \rho)^2 \qquad (8.5)$$

according to equation (8.3) which is applied to the ions. No longitudinal velocities $\boldsymbol{v}_{v\parallel}$ are present. The influence of such motions will be discussed briefly in § 3 of this chapter.

(v) Consider small perturbations of the initial state, where the densities are given by $n_i = N + \tilde{n}_i$, $n_e = N + \tilde{n}_e$, the pressure tensors by $\pi_i = \pi_{i0} + \tilde{\pi}_i$, $\pi_e = \pi_{e0} + \tilde{\pi}_e$, the mass velocities by $\boldsymbol{v}_i = \boldsymbol{v}_{i0} + \tilde{\boldsymbol{v}}_i = \tilde{\boldsymbol{v}}_i$, $\boldsymbol{v}_e = \boldsymbol{v}_{e0} + \tilde{\boldsymbol{v}}_e$ and the electric field by $\boldsymbol{E} = \boldsymbol{E}_0 - \nabla\tilde{\phi}$. All perturbations, which are denoted by a curved bar, are small and their products will be neglected in the present linearized theory.

(vi) Only velocity perturbations in planes perpendicular to the magnetic field are considered here so that $\tilde{\boldsymbol{v}}_i = \tilde{\boldsymbol{v}}_{i\perp}$ and $\tilde{\boldsymbol{v}}_e = \tilde{\boldsymbol{v}}_{e\perp}$.

(vii) The particle density is high enough for the plasma to become electrically quasi-neutral in the sense that $|\tilde{n}_i - \tilde{n}_e| \ll |\tilde{n}_i + \tilde{n}_e|$, and that $\varepsilon_0 \ll Nm_i/B^2$. The approximation $n_i \approx n_e$ can easily be verified by substituting (8.10) into the results of the present chapter.

(viii) The periods of gyration of ions and electrons are much shorter than the characteristic times during which the perturbations change appreciably. Likewise, the Larmor radii should be much smaller than the wave lengths of the perturbation.

With these starting points we substitute (8.5) into (8.4) which becomes

$$n_\nu \boldsymbol{v}_{\nu\perp} = -\frac{2n_\nu m_\nu}{q_\nu B^2}\boldsymbol{B} \times (\boldsymbol{v}_\nu \times \boldsymbol{\Omega}) - n_\nu \nabla\tilde{\phi} \times \boldsymbol{B}/B^2 \tag{8.6}$$

$$+ n_\nu\left(\frac{m_\nu}{q_\nu} - \frac{m_\mathrm{i}}{e}\right)[\boldsymbol{g} + \tfrac{1}{2}\nabla(\boldsymbol{\Omega} \times \boldsymbol{\rho})^2] \times \boldsymbol{B}/B^2$$

$$-\left(\frac{1}{q_\nu}\operatorname{div}\boldsymbol{\pi}_\nu - \frac{n_\nu}{eN}\operatorname{div}\boldsymbol{\pi}_{\mathrm{i}0}\right) \times \boldsymbol{B}/B^2 - n_\nu m_\nu \frac{\partial \boldsymbol{v}_\nu}{\partial t} \times \boldsymbol{B}/q_\nu B^2, \quad \nu=\mathrm{i, e}.$$

Unperturbed quantities are time-independent and from (8.6) therefore follows that

$$\frac{\partial \boldsymbol{v}_{\nu\perp}}{\partial t} = -\frac{2m_\nu}{q_\nu B^2}\boldsymbol{B} \times \left(\frac{\partial \boldsymbol{v}_\nu}{\partial t} \times \boldsymbol{\Omega}\right) - \nabla\frac{\partial\tilde{\phi}}{\partial t} \times \boldsymbol{B}/B^2$$

$$-\frac{\partial}{\partial t}\left(\frac{1}{n_\nu}\operatorname{div}\boldsymbol{\pi}_\nu\right) \times \boldsymbol{B}/q_\nu B^2 - m_\nu\frac{\partial^2 \boldsymbol{v}_\nu}{\partial t^2} \times \boldsymbol{B}/q_\nu B^2 \tag{8.7}$$

and

$$\frac{\partial \boldsymbol{v}_{\nu\perp}}{\partial t} \times \boldsymbol{B} = \nabla_\perp\frac{\partial\tilde{\phi}}{\partial t} + \frac{\partial}{\partial t}\left(\frac{1}{n_\nu q_\nu}\operatorname{div}\boldsymbol{\pi}_\nu\right)_\perp$$

$$+\frac{m_\nu}{q_\nu}\frac{\partial^2 \boldsymbol{v}_{\nu\perp}}{\partial t^2} - \frac{2m_\nu}{q_\nu}\left(\frac{\partial \boldsymbol{v}_\nu}{\partial t} \times \boldsymbol{\Omega}\right)_\perp. \tag{8.8}$$

Now take the divergence of (8.6) where the left hand member is connected with the density changes by (5.17) and $\boldsymbol{v}_{\nu\|} = 0$. From well-known vector identities, and with the assumptions of (iv) in mind, we obtain in analogy with equation (5.62):

$$-\frac{\partial n_\nu}{\partial t} = \operatorname{div}(n_\nu \boldsymbol{v}_\nu) = -2\operatorname{div}\left[\frac{n_\nu}{\omega_\nu B}\boldsymbol{B} \times (\boldsymbol{v}_\nu \times \boldsymbol{\Omega})\right]$$

$$-(\nabla\tilde{\phi} \times \boldsymbol{B}) \cdot \nabla(n_\nu/B^2) + \frac{1}{B}\left(\frac{1}{\omega_\nu} - \frac{1}{\omega_\mathrm{i}}\right)\{[\boldsymbol{g} + \tfrac{1}{2}\nabla(\boldsymbol{\Omega} \times \boldsymbol{\rho})^2] \times \boldsymbol{B}\} \cdot \nabla n_\nu$$

$$+\frac{2}{q_\nu B^3}(\boldsymbol{B} \times \nabla B) \cdot \operatorname{div}\boldsymbol{\pi}_\nu - \frac{1}{q_\nu B^2}\boldsymbol{B} \cdot \operatorname{curl}(\operatorname{div}\boldsymbol{\pi}_\nu)$$

$$-\frac{1}{eNB^2}(\boldsymbol{B} \times \operatorname{div}\boldsymbol{\pi}_{\mathrm{i}0}) \cdot \nabla n_\nu - \operatorname{div}\left\{\frac{N}{\omega_\nu B}\left[\nabla_\perp\frac{\partial\tilde{\phi}}{\partial t} + \frac{\partial}{\partial t}\left(\frac{1}{n_\nu q_\nu}\operatorname{div}\boldsymbol{\pi}_\nu\right)_\perp\right]\right\}$$

$$-\operatorname{div}\left\{\frac{N}{\omega_\nu^2}\left[\frac{\partial^2 \boldsymbol{v}_{\nu\perp}}{\partial t^2} - 2\left(\frac{\partial \boldsymbol{v}_\nu}{\partial t} \times \boldsymbol{\Omega}\right)_\perp\right]\right\}, \tag{8.9}$$

where we have introduced the gyro frequencies $\omega_\nu = q_\nu B/m_\nu$, including the sign of the charge.

To the linearized equation (8.9) which contains $\tilde{n}_i$, $\tilde{n}_e$, $\tilde{\phi}$, $\tilde{\boldsymbol{v}}_i$ and $\tilde{\boldsymbol{v}}_e$ we shall add Poisson's equation (2.7) for the perturbed electric potential:

$$\tilde{n}_i - \tilde{n}_e = -(\varepsilon_0/e)\nabla^2 \tilde{\phi}. \tag{8.10}$$

In many situations of practical interest even a small deviation $\tilde{n}_i - \tilde{n}_e$ from electric neutrality produces excessively large electric fields, which cannot be created by the limited amount of energy stored in the plasma. This occurs e.g. when the characteristic macroscopic dimensions are much larger than the Debye length, $(\varepsilon_0 kT_\nu/e^2 n)^{\frac{1}{2}}$, where T_ν is the temperature. The plasma then has to be quasi-neutral in the sense defined in (vii) of this paragraph. On the other hand, when the plasma density is very low, the deviation $|n_i - n_e|/|n_i + n_e|$ may approach unity. With the exception of § 2.3 of this chapter such a situation is ruled out in the treatment which follows.

Before turning to the special applications of these results we shall discuss the physical significance of the different terms in the last member of equation (8.9). The third term contains $\boldsymbol{g}$ and $\boldsymbol{\Omega} \times \boldsymbol{\rho}$. Its coefficient ahead of ∇n_ν is the drift velocity of the particles due to the gravitation and centrifugal fields. The form is analogous to that deduced earlier in Ch. 5, § 2.3 for the drift of density distributions. The fourth term contains ∇B and represents the magnetic gradient drift. Both the third and fourth terms are different for ions and electrons and produce a charge separation. The latter is due to the relative motion between the disturbed density distributions of ions and electrons.

In the second term are represented two effects of the electric field and the potential $\tilde{\phi}$ which arise from the charge separation. Firstly, a contribution $-(\nabla\tilde{\phi} \times \boldsymbol{B}/B^2) \cdot \nabla n_\nu$ is included which generates a convection of surfaces of constant density at the electric drift velocity $\boldsymbol{u}_E = \boldsymbol{E} \times \boldsymbol{B}/B^2$. This velocity is equal for ions and electrons. Secondly, in the case of an inhomogeneous field, there is also a compression of the ion and the electron gases at the rate $2(n_\nu/B^3)(\nabla\tilde{\phi} \times \boldsymbol{B}) \cdot \nabla B = n_\nu \operatorname{div} \boldsymbol{u}_E$ when particles are forced to drift across the surfaces $B = \text{const.}$ at the velocity $\boldsymbol{u}_E$. This latter effect has been discussed in Ch. 6, § 1 in connexion with magnetic compression phenomena.

A further influence of the electric field comes from the first contribution in the seventh term which includes $\partial\tilde{\phi}/\partial t$. It is due to a deviation from the zero order orbits being caused by the inertia drift of ions according to

equation (3.25). As a result, the ions will be displaced somewhat with respect to the electrons, and this produces an electric polarization. The latter can sometimes be described in terms of an equivalent dielectric constant, as shown in equation (3.51) and in Ch. 5, § 2.3.

The main contribution from the Coriolis force comes from the first term. It differs for ions and electrons, and produces additional charge separation effects. According to Ch. 7, § 2.2 the motion in a frame rotating at the constant angular velocity $\boldsymbol{\Omega}$ becomes equivalent to the motion in a stationary frame where ions and electrons "feel" the equivalent magnetic fields $\boldsymbol{B} + 2m_i\boldsymbol{\Omega}/e$ and $\boldsymbol{B} - 2m_e\boldsymbol{\Omega}/e$. Thus, the equivalent electric field drifts of ions and electrons are no longer equal in such a frame.

The total flux of particles through a surface and its changes in time are not only given by the guiding centre motion, but also by the gyration which is represented by the contribution from the pressure tensor in the fifth, sixth and seventh terms of the right hand member of equation (8.9). Finite Larmor radius effects associated with higher order contributions to the pressure tensor arise mainly from the fifth term of the same member. They will be treated in § 2.5. Similar effects are also due to the unperturbed tensor in the sixth term.

Finally, the last term of (8.9) includes some higher order contributions from the inertia force on ions. It can usually be neglected.

2.3. GRAVITATION INSTABILITY OF A SHARP PLASMA BOUNDARY AT SMALL LARMOR RADII

In Figure 8.5a a sharp boundary is assumed which in the unperturbed state separates a plasma of density $N = N_1$ in the upper half plane from a plasma of density $N = N_2$ in the lower half plane. Both N_1 and N_2 are constant. A gravitation force $\boldsymbol{g}$ exists along the positive or negative y direction and the plasma is supported by a homogeneous magnetic field directed out of the plane of the figure. For the sake of simplicity we assume here that the thermal energies of ions and electrons are very small, so that all finite Larmor radius effects can be neglected.

Suppose now that the boundary between the upper and lower regions is perturbed sinusoidally at a wave length $2\pi/k_x$ and that the perturbed state initially is electrically neutral with the constant densities N_1 and N_2 on either side of the boundary. Thus, the amplitudes λ_i and λ_e of the spatial perturbations of the ion and electron clouds are

$$\lambda_i(0) = \lambda_0 \sin k_x x = \lambda_e(0) \tag{8.11}$$

in the initial state.

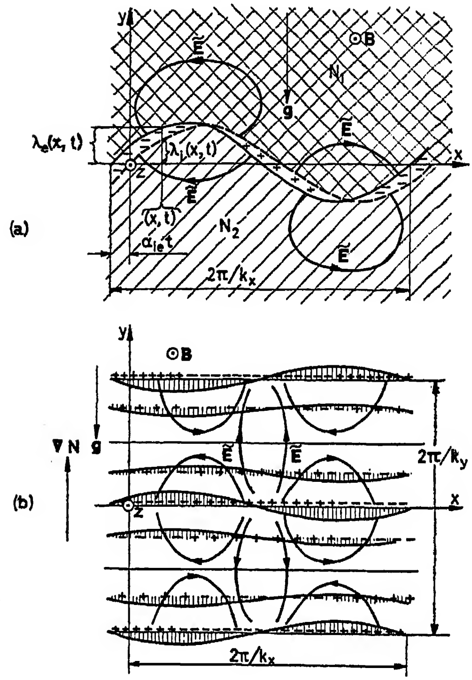


Fig. 8.5. Charge separation in the interior of a magnetized plasma due to gravitation drifts of density perturbations. (a) Sharp boundary between two regions of densities N_1 and N_2. (b) Continuous unperturbed density gradient, $\mathrm{d}N/\mathrm{d}y$, with sinusoidal density perturbations in the x and y directions forms a "bumpy" pattern of surfaces of constant density. The vertically shaded areas indicate excess of density when deviation from horizontal line is downwards and deficit of density when it is upwards.

For the internal regions of the plasma, with the exception of the boundary, equation (8.9) reduces to

$$-\frac{\partial n_i}{\partial t} = -\left(\nabla\tilde{\phi} \times \boldsymbol{B}/B^2\right)\cdot \nabla n_i - \frac{N}{\omega_i B}\nabla^2\frac{\partial \tilde{\phi}}{\partial t} \tag{8.12}$$

and

$$-\frac{\partial n_e}{\partial t} = -\left(\nabla\tilde{\phi} \times \boldsymbol{B}/B^2\right)\cdot \nabla n_e - \frac{1}{\omega_i B}(\boldsymbol{g} \times \boldsymbol{B})\cdot \nabla n_e. \tag{8.13}$$

Since the unperturbed densities N_1 and N_2 are constant the linearized equation (8.13) yields

$$\left[\frac{\partial}{\partial t} - \frac{1}{\omega_i B}(\boldsymbol{g} \times \boldsymbol{B})\cdot \nabla\right] n_e = 0 \tag{8.14}$$

which shows that the electron density distribution is constant in a frame which moves at the velocity $\boldsymbol{u}_g = -\boldsymbol{g} \times \boldsymbol{B}/\omega_i B$ with respect to the centre of mass of the unperturbed state. In the initial state we have $n_e(0)$ equal to N_1 and N_2 in the upper and lower regions, respectively. We therefore conclude that n_e remains constant inside the plasma at all later times. This is so, because the inertia drift of electrons has been neglected.

The difference between equations (8.12) and (8.13) can be combined with (8.10) from which follows that

$$\frac{1}{e}\left(\varepsilon_0 + \frac{m_i N}{B^2}\right) \nabla^2 \frac{\partial \tilde{\phi}}{\partial t} = \frac{1}{\omega_i B} (\boldsymbol{g} \times \boldsymbol{B}) \cdot \nabla n_e. \tag{8.15}$$

Observing that n_e is constant inside the plasma and that $\tilde{\phi}$ vanishes in the initial, neutral state, we now obtain

$$\nabla^2 \tilde{\phi} = 0 \quad \text{(inside plasma).} \tag{8.16}$$

Due to the continuity of the electric potential across the boundary the solution of this equation becomes

$$\tilde{\phi} = [S_b(t) \sin k_x x + C_b(t) \cos k_x x] \exp(\mp k_x y) \tag{8.17}$$

which is finite for $y = \pm\infty$. Here S_b and C_b are functions of time, $2\pi/k_x$ is the wave length of the disturbance at the perturbed boundary and the minus and plus signs refer to the upper and lower half planes of Figure 8.5a.

According to the orbit theory and to equations (8.12) and (8.13) the guiding centre drifts of ions and electrons are in first order equal to the drifts of the corresponding density distributions. The vertical drift velocity is $(1/B)\partial\tilde{\phi}/\partial x$ for both ions and electrons. Further, the horizontal drift of the ions is nearly zero and the electrons move along the x axis with the velocity $u_g = -g/\omega_i$, where g can have either sign. In the coordinate systems following the particle drifts we should therefore observe spatial amplitudes which change at the rate

$$\frac{\partial}{\partial t}\lambda_i = \left(\frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x}\right)\lambda_e = \frac{1}{B}\left(\frac{\partial \tilde{\phi}}{\partial x}\right)_{y=0}, \tag{8.18}$$

where the right hand member is evaluated at the boundary which is only slightly perturbed. For the amplitudes of the spatial perturbations we further assume the forms

$$\lambda_i = S_i(t) \sin k_x x + C_i(t) \cos k_x x \tag{8.19}$$

and

$$\lambda_e = S_e(t) \sin(k_x x - \alpha_{ie} t) + C_e(t) \cos(k_x x - \alpha_{ie} t), \tag{8.20}$$

where $\alpha_{ie} = k_x u_g$ is the frequency at which the perturbations of the electron cloud are seen to pass the ion cloud.

The separation of the ion and electron clouds by the drift motions produces an electric charge

$$\sigma_s = e(N_1 - N_2)(\lambda_i - \lambda_e) \tag{8.21}$$

per unit area of the xz plane. Since the perturbation is small this can be considered as a surface charge located closely to the undisturbed boundary $y = 0$. Its rate of change follows from

$$-\frac{\partial \sigma_s}{\partial t} = e(N_2 - N_1)u_g \frac{\partial \lambda_e}{\partial x} \tag{8.22}$$

when use is made of equation (8.18). The change is determined by the rate at which the electron cloud "sweeps" across the ion cloud in the x direction.

Observe that the motion of the electron cloud is undistorted by inertia drifts and that the rate of change of the charge density produced by the electrons is given purely by the convective motion u_g. The ion cloud, on the other hand, moves only slowly by the inertia drift, but is "piled up" by the latter in such a way that an additional contribution to the space charges is generated.

We assume for a moment that the density gradients are finite but very large at the boundary. Then, the rate of change of the charge by convection of electrons is given by the right hand member of equation (8.15). This expression tends to the right hand member of (8.22) in the limit of a sharp boundary. We therefore conclude that the present problem can be treated as that of a surface charge given by (8.21) and situated between two equivalent dielectric media with the dielectric constants $\varepsilon_0 + m_i N_1/B^2$ and $\varepsilon_0 + m_i N_2/B^2$. This is only true as long as we can neglect the influence from the Larmor motion of ions and electrons (cf. the discussion in connexion with (5.55)). The present result holds also when condition (vii) of § 2.2 is no longer applicable.

According to electrostatic theory the normal components of the electric field $-\nabla\tilde{\phi}$ on each side of the charged layer will be related by

$$\sigma_s = -(\varepsilon_0 + m_i N_1/B^2)\left(\frac{\partial \tilde{\phi}}{\partial y}\right)_1 + (\varepsilon_0 + m_i N_2/B^2)\left(\frac{\partial \tilde{\phi}}{\partial y}\right)_2, \tag{8.23}$$

where the derivatives are evaluated at each side and close to the boundary. In combination with the solution (8.17) we obtain

$$-\left(\frac{\partial\tilde{\phi}}{\partial y}\right)_1 = \left(\frac{\partial\tilde{\phi}}{\partial y}\right)_2 = \sigma_s/[2\varepsilon_0 + m_i(N_1 + N_2)/B^2]. \qquad (8.24)$$

From combination of equations (8.18), (8.21) and the partial derivative of (8.24) with respect to x we have

$$\dot{S}_i \sin k_x x + \dot{C}_i \cos k_x x = \dot{S}_e \sin(k_x x - \alpha_{ie}t) + \dot{C}_e \cos(k_x x - \alpha_{ie}t)$$
$$= \tfrac{1}{4}\alpha_{ie}\Gamma[S_i \cos k_x x - C_i \sin k_x x - S_e \cos(k_x x - \alpha_{ie}t) + C_e \sin(k_x x - \alpha_{ie}t)] \qquad (8.25)$$

with

$$\Gamma = -\frac{4e^2(N_1 - N_2)}{k_x g m_i[2\varepsilon_0 + m_i(N_1 + N_2)/B^2]}. \qquad (8.26)$$

The present equations can be solved by elementary methods and for the detailed deductions reference is made to earlier investigations (LEHNERT [1961]). After development of the trigonometrical functions and rearrangement of the equations in terms of $S_i + S_e$, $S_i - S_e$, $C_i + C_e$ and $C_i - C_e$ the result becomes $S_i = S_e \equiv \lambda_s$ and $C_i = -C_e \equiv \lambda_c$ where

$$\lambda_s/\lambda_0 = \cos(\tfrac{1}{2}\alpha_{ie}t)\cosh(\tfrac{1}{2}\Gamma'\alpha_{ie}t) + (1/\Gamma')\sin(\tfrac{1}{2}\alpha_{ie}t)\sinh(\tfrac{1}{2}\Gamma'\alpha_{ie}t), \qquad \Gamma > 1, \qquad (8.27)$$

$$\lambda_s/\lambda_0 = \cos(\tfrac{1}{2}\alpha_{ie}t)\cos(\tfrac{1}{2}|\Gamma'|\alpha_{ie}t) + |1/\Gamma'|\sin(\tfrac{1}{2}\alpha_{ie}t)\sin(\tfrac{1}{2}|\Gamma'|\alpha_{ie}t), \qquad \Gamma < 1, \qquad (8.28)$$

and $\Gamma' = (\Gamma - 1)^{\frac{1}{2}}$. Similar relations are valid for λ_c.

The results (8.26), (8.27) and (8.28) show that the system is unstable when $\Gamma > 1$ and stable when $\Gamma < 1$. The sign of Γ is given by the sign of $(N_1 - N_2)/g$ according to equation (8.26). There are three possibilities:

(i) If $(N_1 - N_2)/g > 0$, i.e. $\Gamma < 1$, we have a light fluid on top of a heavy one. This system is always stable.

(ii) If $\Gamma > 1$ and $(N_1 - N_2)/g < 0$ we have a heavy fluid on top of a lighter one. The system is unstable. This always occurs for sufficiently large values of $(N_1 - N_2)/(N_1 + N_2)$. In the limit where N_2 tends to zero and the boundary divides the plasma from vacuum as in Figure 8.1 the solution (8.27) becomes

$$\lambda_s = \lambda_0 \exp(gk_x t^2)^{\frac{1}{2}} \qquad (8.29)$$

for small separations $\alpha_{ie}t$ (see Figure 8.5a). This result was earlier found by

ROSENBLUTH and LONGMIRE [1957]. The plasma then "explodes" at an exponentially increasing rate across the magnetic field. When the amplitude has increased far enough its growth rate will be limited by non-linear effects.

(iii) If $0 < \Gamma < 1$ we will still have a heavy fluid on top of a lighter one and $(N_1 - N_2)/g < 0$. In this case the growth also starts with an increasing amplitude at small separations $\alpha_{ie}t$, as in case (ii). However, the driving force from the density difference $N_1 - N_2$ which produces the electric field is now smaller and the growth slower. When the separation has reached $\alpha_{ie}t = \frac{1}{2}\pi$, the electric field drift is reversed and oscillations are produced. Such oscillations also occur in weakly unstable situations in case (ii), but in the present case the amplitude remains limited as shown by equation (8.28)

It should be noticed that the solution in the limit of (8.29) is independent of the magnetic field and the growth rate is exactly the same as that found for the hydrodynamic Rayleigh-Taylor instability. In the more general cases of equations (8.27) and (8.28), however, the solution depends on the magnetic field strength.

2.4. THE EFFECT OF A FINITE DENSITY GRADIENT

In connexion with Figure 8.5a we have just seen that a sufficiently small density jump at an inner plasma boundary gives driving forces which are small enough for the boundary to remain stable against flutes of a given wave length. A real plasma never has a sharp boundary and we are therefore lead to discuss what happens in presence of a finite density gradient. In particular, it should be remembered that equilibrium configurations in which a sheet current separates a field-free fluid from a vacuum magnetic field have stability properties of entirely local character. The situation is quite different for a configuration of mixed fluid and magnetic field with distributed volume current and finite pressure and density gradients (cf. BERKOWITZ *et al.* [1958]).

Assume the unperturbed density distribution to be given by a gradient $N' \equiv \mathrm{d}N/\mathrm{d}y$ as demonstrated in Figure 8.5b. We still consider gravitation instability in a homogeneous magnetic field $\boldsymbol{B}$, but shall now include effects of the thermal motion and of a finite Larmor radius. On the other hand, we restrict the treatment to characteristic lengths $L_{cN} \equiv N/|\mathrm{d}N/\mathrm{d}y|$ of the unperturbed density distribution which are much larger than the length $\tilde{L}_{cmin}$ of steepest spatial variation of the perturbation. Finally assume the velocities of gyration W_i and W_e of ions and electrons to be constant over space in the unperturbed state.

Density perturbations are now imposed on this state which are sinusoidal in the x and y directions with wave lengths $L_x = 2\pi/k_x$ and $L_y = 2\pi/k_y$. The perturbed surfaces of constant density will then become deformed a little from their unperturbed horizontal positions as indicated in Figure 8.5b. This produces a sinusoidal "bumpy" pattern superimposed on the unperturbed density distribution and gives rise to space charges and a perturbed electric field distribution $\tilde{\boldsymbol{E}}$, analogous to that of the sharp boundary case of Figure 8.5a.

According to the discussions in Ch. 4, § 1.2 and in § 3, equation (4.110), we assume the equivalent magnetic moment $M = mW^2/2B$ to be constant for ions and electrons. Since B is constant in the present problem also W_i^2 and W_e^2 remain at their constant unperturbed values. Later in § 2.5 we shall discuss the influence from higher order contributions to the pressure tensor. Here we restrict ourselves to the case where the latter can be approximated by scalar pressures and write

$$(\mathrm{div}\, \boldsymbol{\pi}_i)_\perp \approx K_{i\perp} \nabla_\perp n_i, \quad (\mathrm{div}\, \boldsymbol{\pi}_e)_\perp \approx K_{e\perp} \nabla_\perp n_e, \tag{8.30}$$

according to equation (5.24). As we shall see in § 2.5 this approximation is justified as long as we restrict ourselves to flat density distributions, where $L_{cN} \gg \tilde{L}_{c\min} = 1/2\pi(k_x^2 + k_y^2)^{\frac{1}{2}}$. In the present case the magnetic field lines are straight and the energies $K_{i\perp} = \overline{M}_i B$ and $K_{e\perp} = \overline{M}_e B$ remain constant to lowest order.

Equation (8.9) now reduces to

$$-\frac{\partial n_v}{\partial t} = -(\nabla \tilde{\phi} \times \boldsymbol{B}/B^2) \cdot \nabla n_v + \frac{1}{B}\left(\frac{1}{\omega_v} - \frac{1}{\omega_i}\right)(\boldsymbol{g} \times \boldsymbol{B}) \cdot \nabla n_v$$
$$+ \frac{K_{i\perp}}{eNB^2} \cdot (\nabla N \times \boldsymbol{B}) \cdot \nabla n_v - \mathrm{div}\left[\frac{N}{\omega_v B}\left[\nabla \frac{\partial \tilde{\phi}}{\partial t} + \frac{\partial}{\partial t}\left(\frac{K_{v\perp}}{n_v q_v} \nabla n_v\right)\right]\right]. \tag{8.31}$$

Linearize this expression and neglect contributions of the order $(L_x^{-2} + L_y^{-2})^{-1}/L_{cN}^2$ compared to unity in the last term of equation (8.31). The result then becomes

$$-\frac{\partial \tilde{n}_i}{\partial t} = \frac{N'}{B} \frac{\partial \tilde{\phi}}{\partial x} + u_f \frac{\partial \tilde{n}_i}{\partial x} - \frac{N}{\omega_i B} \nabla^2 \frac{\partial \tilde{\phi}}{\partial t} - \frac{K_{i\perp}}{e\omega_i B} \nabla^2 \frac{\partial \tilde{n}_i}{\partial t} \tag{8.32}$$

for ions and

$$-\frac{\partial \tilde{n}_e}{\partial t} = \frac{N'}{B} \frac{\partial \tilde{\phi}}{\partial x} + (u_f + u_g) \frac{\partial \tilde{n}_e}{\partial x} \tag{8.33}$$

for electrons, where $u_f = N'K_{i\perp}/eNB$ and $u_g = -g/\omega_i$. Observe that the last term of (8.32) arises from a contribution by $\boldsymbol{v}_g$ in (5.53).

For $(L_x^{-2} + L_y^{-2})^{-1}/L_{cN}^2 \ll 1$ and $\varepsilon_0 \ll Nm_i/B^2$ equations (8.32), (8.33) and (8.10) can be combined to

$$\left[\nabla^2 \frac{\partial^2}{\partial t^2} + (u_g + u_f - u_K)\nabla^2 \frac{\partial^2}{\partial x \partial t} + G_{ie}\frac{\partial^2}{\partial x^2}\right](\tilde{\phi}, \tilde{n}_i, \tilde{n}_e) = 0, \tag{8.34}$$

where $u_K = K_{i\perp}N'/Nm_i\omega_i$ and $G_{ie} = -u_g\,\omega_i N'/N$. We have here assumed that the terms containing $\tilde{n}_i$, $\tilde{n}_e$ and $\tilde{\phi}$ in equations (8.32) and (8.33) do not vanish separately. Otherwise the system would become overdetermined and could possibly be satisfied only under very restricted conditions.

In the deduction of (8.34) we have neglected ε_0 compared to Nm_i/B^2, which is usually a good approximation. It implies that the Alfvén velocity in the medium is much smaller than that of light. The "true" electric charge $e(n_i - n_e)$ is then very small. The charge generated by the gravitation drift, is mainly represented by the right hand member of equation (8.15). It is balanced by a charge arising from the inertia drift of the ions, as given by the contribution from m_iN/B^2 in the left hand member of the same equation. Due to this effect the plasma behaves like a polarizable medium with a high dielectric constant, but generally there are also other charge separation effects involved as will be shown later in § 2.5.

For normal modes of the form $\exp\,[i(k_xx + k_yy + \omega t)]$ equation (8.34) immediately gives the dispersion relation (LEHNERT [1961], ROBERTS and TAYLOR [1962])

$$\begin{aligned}\omega &= \tfrac{1}{2}(\alpha_{ie} - k_xu_f + k_xu_K) \pm \tfrac{1}{2}[(\alpha_{ie} - k_xu_f + k_xu_K)^2 - 4\alpha_{ie}{}^2\Gamma]^{\frac{1}{2}} \\ &= \tfrac{1}{2}\alpha_{ie} \pm \tfrac{1}{2}\alpha_{ie}(1 - \Gamma)^{\frac{1}{2}}\end{aligned} \tag{8.35}$$

with $\alpha_{ie} = -k_xu_g$ and

$$\Gamma = -\frac{4N'\omega_i^2}{N(k_x^2 + k_y^2)g}. \tag{8.36}$$

The last member of (8.35) is obtained because $u_f \equiv u_K$. In fact, we have used separate notations for u_f and u_K only to be able to examine how different physical effects enter the problem. The contribution u_K comes from the last term of (8.32). We shall discuss this further in the refined theory of § 2.5 of which (8.35) is a special case valid for moderately large density gradients.

The obtained result for ω apparently depends on y and one might therefore ask whether it is in fact a solution of (8.34) or not, i.e. if a method based on localized perturbations is applicable here. However, since L_{cN} is much larger than $\tilde{L}_{c\min}$ the expressions (8.35) and (8.36) for ω will become approximately independent of y for arbitrary distributions $N(y)$. Moreover, in the particular case where $N =$ const. exp. (const. y) the dispersion relation for ω does not depend on y and becomes an exact solution of (8.34). Thus, in a theory on localized perturbations we can always choose such functions N which yield at least approximate solutions of the form (8.35) within a large region of space, and which represent the essential physical properties of our problem.

Equations (8.35) and (8.36) yield the stability condition $\Gamma < 1$ which is closely related to the results (8.27) and (8.28) for a sharp plasma boundary. Thus, there is stability for all modes when $N'/g > 0$ and the density gradient and the gravitation force act in the same direction, as in case (i) of § 2.3. For $\Gamma > 1$ there is instability since the driving forces are too large; this corresponds to case (ii) of § 2.3. Finally, when $0 < \Gamma < 1$ and $N'/g < 0$ the destabilizing effects are suppressed on account of the small driving forces of the flat density distribution and of the scrambling mechanism due to the reversals of the electric field as demonstrated in case (iii) of § 2.3. Then, the density disturbances and the perturbed electric field oscillate with limited amplitudes.

In the present example finite Larmor radius effects appear in the terms $u_\mathfrak{f}$ and u_K which arise from the sixth and seventh terms of the right hand member of equation (8.9), respectively. With the assumption of scalar pressures in (8.30) these contributions cancel since $u_\mathfrak{f} \equiv u_K$. The stabilization which is found here can be considered as an effect of the finite density gradient in (8.36) where $|\Gamma|$ is seen to decrease with $|N'|$. However, it may equally well be considered as a result of the Hall effect. This is easily seen if equations (8.3) for ions and electrons are combined to the generalized form of Ohm's law (cf. ROBERTS and TAYLOR [1962]). A study of the Hall effect in connexion with plasma stability is also due to WARE [1961] and to TAYLER [1962].

2.5. FINITE LARMOR RADIUS EFFECTS FROM HIGHER ORDER CONTRIBUTIONS TO THE PRESSURE TENSOR

When we no longer impose the restriction that the characteristic length L_{cN} of the density distribution should be much longer than the wave length $\tilde{L}_{c\min}$ of steepest variation of the perturbation, the situation becomes altered.

In particular, this requires higher order approximations to be taken into account in the pressure tensor and the fifth term of the right hand member of (8.9) should be included. This leads to a finite Larmor radius effect and a charge separation phenomenon in addition to those involved in § 2.4. It was first treated by ROSENBLUTH *et al.* [1962] in terms of the Boltzmann equation and is connected with the term $(1/4)a^2\nabla_\perp^2 F$ in equation (3.40). Here the stability criterion found by the latter authors will be deduced from the theory of § 2.2, including terms up to second order, as compared to the second member of equation (8.9).

For the sake of simplicity, we put $N'/N =$ const. and assume the y dependence of the perturbation to be negligible, so that the flute disturbance has the form of long "spokes", extended in the y direction as demonstrated

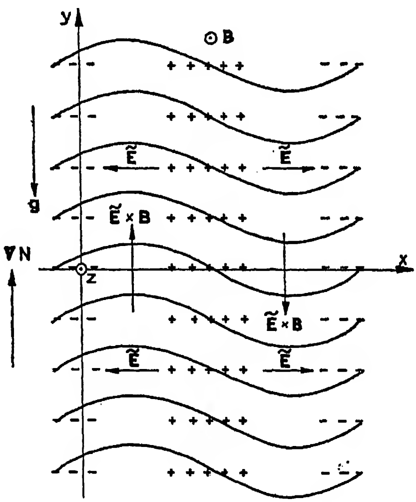


Fig. 8.6. Conditions similar to that of Figure 8.5b, but the density perturbation has the form of long "spokes", extended in the y direction. Surfaces of constant density in initial state are given by sinusoidal lines.

in Figure 8.6. We still treat the gravitation instability earlier discussed in § 2.4. In equation (8.32) we then only have to put the derivatives of the perturbation with respect to y equal to zero and further use the more general expressions (5.48), (5.49) and (5.50) for the pressure tensor. This yields

$$(\operatorname{div} \pi_i)_\perp =$$

$$= \nabla(K_{i\perp} n_i) + \tfrac{1}{4}\, eB\bar{a}_i^2 \left(-N \frac{\partial^2 v_{iy}}{\partial x^2} + N' \frac{\partial v_{ix}}{\partial x},\; N' \frac{\partial v_{iy}}{\partial x} + N \frac{\partial^2 v_{ix}}{\partial x^2},\; 0 \right). \qquad (8.37)$$

Since the Larmor radius of electrons is small we still assume the electron pressure tensor to be given by (8.30). Consequently,

$$\frac{1}{eB^2}\boldsymbol{B}\cdot\operatorname{curl}\left(\operatorname{div}\pi_i\right) = \tfrac{1}{2}\bar{a}_i^2\,\frac{N'}{B}\,\frac{\partial^3\tilde{\phi}}{\partial x^3} \tag{8.38}$$

in second order. The result of this is that equation (8.33) of § 2.4 remains the same and that the term (8.38) has to be subtracted from the right hand member of (8.32).

In analogy with (8.34) and in the first approximation we obtain with $u_f = N'K_{i\perp}/eNB$

$$\left[\frac{\partial^2}{\partial t^2} + (u_g + u_f)\frac{\partial^2}{\partial x\partial t} + G_{ie}\right]\frac{\partial^2}{\partial x^2}\left(\tilde{\phi}, \tilde{n}_i, \tilde{n}_e\right) = 0, \tag{8.39}$$

where $G_{ie} = -u_g\,\omega_i N'/N$ as in § 2.4. For normal modes of the form $\exp\,[i(k_x x + \omega t)]$ and with $\alpha_{ie}=k_x g/u_i$ the dispersion relation is

$$\omega = \tfrac{1}{2}(\alpha_{ie} - k_x u_f)\left\{1 \pm \left[1 - \frac{\Gamma}{(1 - k_x u_f/\alpha_{ie})^2}\right]^{\frac{1}{2}}\right\}, \tag{8.40}$$

where Γ is given by equation (8.36) for $k_y = 0$. A direct solution of (8.4) without iteration yields a cubic relation with two roots corresponding to (8.40) and one root of order ω_i. The latter is irrelevant since adiabatic invariance has been assumed. We further observe that steep density gradients are allowed to exist here and the present localized perturbation analysis is therefore correct only when $N'/N =$ const.

Equation (8.40) agrees with a relation found by ROSENBLUTH *et al.* [1962] and has also been derived from a single-fluid model by ROBERTS and TAYLOR [1962]. The stability criterion now becomes

$$\Gamma < \left(1 - \frac{k_x u_f}{\alpha_{ie}}\right)^2 \text{ or } \left(\frac{k_x \bar{a}_i^2 N'}{N}\right)^2\left(1 - \frac{\alpha_{ie}}{k_x u_f}\right)^2 > 16\left(-\frac{gN'}{N\omega_i^2}\right) \tag{8.41}$$

where

$$k_x u_f/\alpha_{ie} = \tfrac{1}{2}\bar{a}_i^2\,\omega_i^2\,\frac{N'}{Ng} = \tfrac{1}{2}W_i^2\,\frac{N'}{Ng}. \tag{8.42}$$

When the Larmor energy $\frac{1}{2}m_i W_i^2$ is small compared to the gravitation work $m_i g|N/N'|$ along the characteristic length $L_{cN} = |N/N'|$ the modifications introduced by higher order contributions to the ion pressure tensor can be neglected and the result becomes identical with that earlier found in § 2.4 for $k_y = 0$.

On the other hand, when the Larmor energy of ions becomes much larger

than the gravitation work performed along L_{cN} the stability is mainly determined by the contribution $k_x u_f/\alpha_{ie}$ in the first of conditions (8.41). The system is then stabilized for a perturbation of a certain wave length, provided that the density gradient and the Larmor radius become large enough to satisfy conditions (8.41).

We observe that the finite density gradient effect represented by α_{ie} and the finite Larmor radius effect of the ion pressure tensor represented by $k_x u_f$ cooperate towards stability for a system where gravity acts in the opposite direction of the density gradient. The former effect determines the stability at flat gradients and the latter at steep gradients of the density.

At this stage attention should be called to a curiosity involved in the present results. If we start with the tensor of (8.30) given by scalar pressures and for a moment discard the term u_K in (8.35) a stability condition would be obtained which happens to be formally identical with the relations (8.41) found by ROSENBLUTH *et al.* [1962]. To neglect u_K implies that we neglect the last term in (8.32) which represents the rate of change of the momentum stored in the Larmor motion. If we, as a next step, keep this term and still assume the expression (8.30) for the pressure tensor to be valid, there is an exact cancellation of the terms containing u_f and u_K in equation (8.35), because $u_f \equiv u_K$. Then, the stability condition changes from relation (8.41) to $\Gamma < 1$ as given in § 2.4 (LEHNERT [1961]). However, this does not mean that the effect found by ROSENBLUTH *et al.* [1962] is cancelled. As shown by ROBERTS and TAYLOR [1962] and by the present analysis the higher order terms of the ion pressure tensor give an additional contribution to the stability condition of exactly the same magnitude as u_f. Therefore we now have $u_f - u_f + u_f = u_f$ in the coefficient in front of $\partial^2/\partial x \partial t$ in equation (8.39). Thus, the stability condition (8.41) is recovered.

The present results give a good illustration to the pitfalls which may arise if terms are discarded only on account of their order in the parameter ε, as they appear in the differential equations. Thus, the ratio between the last and the second terms of the right hand member of (8.32) is of the order of $(m/eB)\ (N/N'\tilde{L}_c\tilde{t}_c)$ where $\tilde{L}_c$ and $\tilde{t}_c$ are the characteristic lengths and times of the perturbation. Likewise the ratio between the term (8.38) and the first term of the right hand member of (8.32) is of the order of $a_i^2/2\tilde{L}_c^2$. At a first sight one could then be tempted to discard the last term of (8.32), i.e. to neglect the contributions from $\boldsymbol{v}_g$ in (5.53) which are due to the momentum changes of the Larmor motion. Likewise one could be tempted to neglect the finite Larmor radius effects represented by the term (8.38). However, such

approximations lead to incorrect results, because the last term of (8.32) and the term (8.38) give rise to contributions in the final results (8.35) and (8.40) which are equally important as those arising from the terms containing $u_f \partial/\partial x$ in equations (8.32) and (8.33). Apparently, not only the order of a certain term is of importance, as it appears in the differential equations, but also its mathematical form.

Finally, the results also demonstrate that a direct substitution of ε_0 in (2.5) by an equivalent dielectric constant, Nm_i/B^2, is a too crude approximation here. With the term (8.38) included the difference between equations (8.32) and (8.33) contains additional pressure terms associated with the Larmor motion which are not included in such an approach. Thus, the resulting equation for the local rate of change $\partial\sigma/\partial t$ of the electric charge density cannot be brought to a form analogous to equations (2.5) and (3.50).

2.6. THE EFFECT OF MAGNETIC COMPRESSION

So far we have only treated the gravitation instability of a plasma in a homogeneous magnetic field. For the case that there is no external gravitation field present but the magnetic field is inhomogeneous, one might expect the gradient of the latter to produce flute instabilities of a similar kind as those arising from the gravitation drift. Thus, a charge separation results from the magnetic gradient drift. Part of this effect is due to a centrifugal acceleration of the longitudinal particle motion along the curved field lines. However, it is not sure that there is a complete analogy between the gravitation instability and the effect of an inhomogeneous field. The latter produces magnetic compression and expansion effects when matter is forced to move across the surfaces $B =$ const. under influence of the electric field drift $\boldsymbol{u}_E = \boldsymbol{E} \times \boldsymbol{B}/B^2$. This will change the stability situation (LEHNERT [1962a]).

The present compression mechanism is directly related to the discussions of Ch. 5, § 2.3 and Ch. 6, §§ 1 and 2.2 which show that the surfaces of constant density do not move with the particle velocity. The corresponding influence on a flute disturbance is demonstrated by the following simple example. Assume a cylindrical configuration like that of Figure 8.2, but with a finite density gradient of the plasma in the radial direction. A wave-shaped distortion of the surfaces of constant density will still produce an electric field $\tilde{\boldsymbol{E}}$ and a corresponding drift motion somewhat as indicated in Figure 8.2. Pressure and density gradients along the field lines are assumed to be rapidly smoothed out so that we can consider the disturbance to be cylin-

drically symmetric with no dependence of the azimuthal coordinate φ. For the displacements of a flute disturbance we then have a situation similar to that of Figure 6.2a and treated in Ch. 6, § 1, (i).

As in the earlier problems of this chapter space charges will arise from the drift motions of ions and electrons along the unperturbed surfaces $N =$ const. In the present situation, however, the motion of the perturbed surfaces of constant density is not given solely by the convection of matter at the velocity $\tilde{\boldsymbol{E}} \times \boldsymbol{B}/B^2$. Additional effects arise from the compression and expansion produced by this velocity during the motion across the surfaces $B =$ const. This occurs at a rate given by div $(\tilde{\boldsymbol{E}} \times \boldsymbol{B}/B^2)$. In other words, local density changes will not only be generated at a rate $(\tilde{\boldsymbol{E}} \times \boldsymbol{B}/B^2) \cdot \nabla N = -(\tilde{E}_z/B)\,(\mathrm{d}N/\mathrm{d}r)$ by convection, but also by magnetic compression and expansion at a rate N div $(\tilde{\boldsymbol{E}} \times \boldsymbol{B}/B^2) = 2N(\tilde{E}_z/B^2)\,(\mathrm{d}B/\mathrm{d}r)$. This occurs because particles are piled up by a non-uniform drift in the radial direction across the surfaces $B =$ const. of Figure 8.2. It is therefore clear that the space charge distribution will be affected by the inhomogeneity of the magnetic field and that this will react on the driving forces of the flute instability. In particular, for a finite density gradient in the outer boundary region of Figure 8.2 we see that the "crest" of a wave-shaped density perturbation will be weakened when it moves out into a weaker magnetic field since $n/B^2 =$ const. according to equation (6.13). The same is true for a "wave-trough" of the perturbation, and we should therefore expect the driving forces to be suppressed by the magnetic field inhomogeneity.

We shall now start a detailed analysis of the present stability problem under the condition that the characteristic length of the unperturbed density distribution is much larger than the wave length $\tilde{L}_{c\min}$ of steepest varation of the perturbations. For the sake of simplicity we study only effects of lowest order and assume the pressure tensor to be given by a scalar and isotropic pressure as in equation (5.25). Before turning to equation (8.9) we consider the relation (5.60) expressing conservation of energy. Without losing the essential features of the present problem we can achieve a considerable simplification by assuming the unperturbed plasma pressures P_ν and the density N to be adiabatically distributed in the sense that $P_\nu =$ const. $N^{5/3}$. The relation between the pressure and density perturbations is then immediately deduced from (5.60) and becomes

$$\tilde{p}_\nu = (5P_\nu/3N)\tilde{n}_\nu \tag{8.43}$$

for all points in space.

Taking the smallness of $\tilde{L}_{c\min}$ into account and using equation (8.43) we can now write (8.9) as

$$-\frac{\partial n_\nu}{\partial t} = \boldsymbol{\varkappa}\cdot\nabla\tilde{\phi} + (\boldsymbol{u}_{B\nu} + \boldsymbol{u}_f)\cdot\nabla n_\nu - \frac{\varepsilon_\nu}{q_\nu}\nabla^2\frac{\partial\tilde{\phi}}{\partial t} - \frac{5}{6}\bar{a}_\nu^2\nabla^2\frac{\partial n_\nu}{\partial t}, \qquad (8.44)$$

where $\varepsilon_\nu = Nm_\nu/B^2$, $\bar{a}_\nu^2 = \overline{W}_\nu^2/\omega_\nu^2$ and

$$\boldsymbol{\varkappa} = \nabla(N/B^2)\times\boldsymbol{B}, \qquad (8.45)$$

$$\boldsymbol{u}_{B\nu} = \frac{10P_\nu}{3Nq_\nu B^3}\boldsymbol{B}\times\nabla B, \qquad (8.46)$$

$$\boldsymbol{u}_f = \nabla P_i\times\boldsymbol{B}/eNB^2. \qquad (8.47)$$

The velocities $\boldsymbol{u}_{B\nu}$ represent mean magnetic gradient drifts. Here scalar products vanish between ∇N, ∇B and ∇P_ν on one hand, and $\boldsymbol{u}_{B\nu}$ and $\boldsymbol{u}_f$ on the other.

In analogy with equations (8.32) and (8.33) and with the coordinates of Figure 8.2 the perturbations now obey the relations

$$-\frac{\partial\tilde{n}_i}{\partial t} = \varkappa_z\frac{\partial\tilde{\phi}}{\partial z} + (u_{Biz} + u_{fz})\frac{\partial\tilde{n}_i}{\partial z} - \frac{\varepsilon_i}{e}\nabla^2\frac{\partial\tilde{\phi}}{\partial t} - \frac{5}{6}\bar{a}_i^2\nabla^2\frac{\partial\tilde{n}_i}{\partial t} \qquad (8.48)$$

for ions and

$$-\frac{\partial\tilde{n}_e}{\partial t} = \varkappa_z\frac{\partial\tilde{\phi}}{\partial z} + (u_{Bez} + u_{fz})\frac{\partial\tilde{n}_e}{\partial z} \qquad (8.49)$$

for electrons. In combination with (8.10) we obtain for $\varepsilon_i \gg \varepsilon_0$

$$\left[\nabla^2\frac{\partial^2}{\partial t^2} - (u_{Biz} - u_{Bez})\nabla^2\frac{\partial^2}{\partial z\partial t} + G_{ie}\frac{\partial^2}{\partial z^2}\right](\tilde{\phi}, \tilde{n}_i, \tilde{n}_e) = 0, \qquad (8.50)$$

where $G_{ie} = -\,\omega_i(B^2/N)\,(u_{Biz} - u_{Bez})\,(\mathrm{d}/\mathrm{d}r)\,(N/B^2)$.

Normal modes of the form $\exp\,[i(k_z z + \omega t)]$ yield the solution

$$\omega = -\tfrac{1}{2}k_z(u_{Biz} - u_{Bez})\,[1 \pm (1 - \Gamma)^{\frac{1}{2}}] \qquad (8.51)$$

with

$$\Gamma = -\frac{4\omega_i B^2(\mathrm{d}/\mathrm{d}r)(N/B^2)}{Nk_z^2(u_{Biz} - u_{Bez})} = \frac{6\omega_i eB^2}{5k_z^2(P_i + P_e)N}\left(\frac{\mathrm{d}N/\mathrm{d}r}{\mathrm{d}B/\mathrm{d}r} - 2\frac{N}{B}\right). \qquad (8.52)$$

Since $\tilde{L}_{c\min}$ is assumed to be much smaller than L_{cN} and $L_{cB} = B/|\mathrm{d}B/\mathrm{d}r| = r$, the expression (8.51) for ω becomes nearly constant within the local region of a wave length $2\pi/k_z$ of the perturbation.

For a very large radial distance r from the axis the present problem degenerates into a plane case. A radial dependence of the form $\exp(ik_r r)$ can then be included with good approximation in the form of the normal modes. This only changes the stability parameter Γ of (8.52) in such a way that k_z^2 is replaced by $(k_r^2 + k_z^2)$, if regions in the vicinity of the symmetry axis are excluded.

The results (8.51), (8.52) and (8.46) show that one can distinguish between three cases as far as stability is concerned:

(i) When $\mathrm{d}N/\mathrm{d}r$ and $\mathrm{d}B/\mathrm{d}r$ have opposite signs the magnetic field bends convexly towards the main body of the plasma as in cusped geometry. The situation is as in Figure 8.4a and at the inner boundary of Figure 8.2. Then Γ is negative and there is always stability for the present perturbations, as is also necessary from energy considerations.

(ii) When $\mathrm{d}N/\mathrm{d}r$ and $\mathrm{d}B/\mathrm{d}r$ have the same sign and $(\mathrm{d}/\mathrm{d}r)\,(N/B^2) > 0$ the magnetic field bends concavely towards the main body of the plasma and the density N varies more rapidly in space than B^2. We then have a situation like that in Figure 8.4b for mirror geometry or like that at the outer boundary of Figure 8.2. For small Larmor radii and small magnetic drift velocities $u_{Biz} - u_{Bez}$ the value of Γ becomes strongly positive and dominates the square root of equation (8.51). The situation is then unstable. However, if the Larmor radii become larger the value of Γ decreases, and for k_z^2 above a certain value the same square root becomes real. The stabilization which results from this is due to the same scrambling mechanism as that discussed in § 2.3 (iii); the only difference is that the drift is produced by the magnetic gradient and not by the gravitation force.

(iii) When $\mathrm{d}N/\mathrm{d}r$ and $\mathrm{d}B/\mathrm{d}r$ have the same sign but $(\mathrm{d}/\mathrm{d}r)\,(N/B^2) < 0$ the magnetic field is still of mirror type like in Figure 8.4b and at the outer boundary of Figure 8.2, but now the gradient ∇B^2 is steeper than ∇N. This yields negative values of Γ and stability is secured for all modes of the present type. The reason for this is as stated at the beginning of this paragraph, namely that a motion of matter across the magnetic field takes place both in the form of convection and of compression or expansion. The destabilizing forces are weakened in the present case. This is also easily seen from the special situation when N/B^2 is constant all over the plasma, when the lowest order solution (6.13) is valid and when $\varkappa$ and Γ vanish identically according to equation (8.45). Then, it does not matter how an element of the fluid moves,

because it always adjusts itself to the local density at the point where it arrives. This implies that the surfaces of constant density remain fixed, whatever motion there may take place in the plasma.

It should be stressed that the result (8.51) is modified by higher order contributions when a more accurate expression is used for the ion pressure tensor, as in § 2.5. However, this does not change the present conclusions (i) – (iii) regarding the magnetic compression effect which is of zero order and works also at vanishingly small Larmor radii.

When a local thermal equilibrium no longer can be established by collisions this simplified analysis breaks down, and the velocity spectrum of particles will change in space and time on account of the magnetic gradient drift.

2.7. THE EFFECT OF CORIOLIS' FORCE

It is obvious that every effect which produces a charge separation will have an influence on the development of a flute instability. A further illustration of this will be given by the effect of Coriolis' force on a rotating plasma. It gives rise to a situation equivalent to that which occurs if ions could move in a magnetic field slightly different from that experienced by the electrons. Consequently, a charge separation will be produced in addition to those arising from external force drifts and inertia drifts.

We shall demonstrate the mechanism in a simple example where a homogeneous magnetic field is directed along the axis of rotation and where the Larmor radius is assumed to be very small. Then, equation (8.9) reduces to

$$
\begin{aligned}
-\Lambda_\nu \frac{\partial n_\nu}{\partial t} &= \Lambda_\nu \operatorname{div}(n_\nu \boldsymbol{v}_\nu) = -[\nabla\tilde{\phi} \times \boldsymbol{B}/B^2] \cdot \nabla n_\nu \\
&+ \tfrac{1}{2}\left(\frac{1}{\omega_\nu} - \frac{1}{\omega_i}\right)[\nabla(\boldsymbol{\Omega} \times \rho)^2 \times \hat{\boldsymbol{B}}] \cdot \nabla n_\nu - \frac{1}{\omega_\nu \Lambda_\nu B} \operatorname{div}\left(N \nabla \frac{\partial \tilde{\phi}}{\partial t}\right),
\end{aligned} \tag{8.53}
$$

where Ω can have either sign and

$$\Lambda_\nu = 1 + 2\Omega/\omega_\nu, \quad \nu = \mathrm{i, e}. \tag{8.54}$$

If we again assume $\varepsilon_i = m_i N/B^2 \gg \varepsilon_0$ and electric quasi-neutrality, and neglect m_e besides m_i equations (8.53) for ions and electrons can easily be combined to

$$\operatorname{div}\left(N\nabla \frac{\partial^2\tilde{\phi}}{\partial t^2}\right) + 2\Omega\Lambda_i(\hat{\boldsymbol{B}} \times \nabla\frac{\partial\tilde{\phi}}{\partial t})\cdot\nabla N$$

$$+ \Omega^2(\rho \times \hat{\boldsymbol{B}})\cdot\nabla\left[\Lambda_i(\hat{\boldsymbol{B}} \times \nabla\tilde{\phi})\cdot\nabla N - \operatorname{div}\left(\frac{N}{\omega_i}\nabla\frac{\partial\tilde{\phi}}{\partial t}\right)\right] = 0. \quad (8.55)$$

Similar differential equations are valid for the perturbations $\tilde{n}_i$ and $\tilde{n}_e$ in the special situation where the characteristic length of N is much larger than the wave lengths of the perturbation. The operators acting on n_ν and $\tilde{\phi}$ in equations (8.53) do not vanish separately; this would otherwise lead to an uninteresting special case.

Study normal modes of the special form $\tilde{\phi} \propto r^n \exp\,[i(m_\varphi\varphi + \omega t)]$, where n is a constant. The dispersion relation then becomes

$$\left(1 - \frac{n^2}{m_\varphi^2} - \frac{nrN'}{Nm_\varphi^2}\right)\omega^2 + \left[\left(1 - \frac{n^2}{m_\varphi^2}\right)m_\varphi\frac{\Omega^2}{\omega_i} + 2\Omega\Lambda_i\frac{rN'}{Nm_\varphi} - n\frac{\Omega^2}{\omega_i}\frac{rN'}{Nm_\varphi}\right]\omega$$

$$- \Omega^2\Lambda_i rN'/N = 0, \quad (8.56)$$

with $N' \equiv dN/dr$. This relation is independent of r if $N = \text{const.}\, r^{\text{const.}}$

In situations of practical interest the angular frequency Ω becomes much smaller than ω_i. The dispersion relation then reduces to

$$\omega\left(1-\frac{n^2}{m_\varphi^2}-n\frac{rN'}{Nm_\varphi^2}\right)/\Omega = -\frac{rN'}{Nm_\varphi} \pm \left[\left(\frac{rN'}{Nm_\varphi}\right)^2 + \frac{rN'}{N}\left(1-\frac{n^2}{m_\varphi^2}-n\frac{rN'}{Nm_\varphi^2}\right)\right]^{\frac{1}{2}}, \quad (8.57)$$

where the first term of the right hand member and the first term inside the square root arise from Coriolis' force and the last term inside the same root is due to the centrifugal force. We shall examine three cases:

(i) $n = 0$. The disturbance has then the form of radial spokes which are independent of r. Equation (8.57) yields stability for such modes when $r > -m_\varphi^2 N/N'$. This is possible at a given value of m_φ for a sufficiently steep density gradient N'. The stabilization is due to Coriolis' force which generates a charge separation counteracting that produced by the centrifugal force.

(ii) $n = m_\varphi$. In this particular case all the terms in (8.57) which are independent of N' will vanish. An instability given by $\omega/\Omega = -1 \pm (1 - m_\varphi)^{\frac{1}{2}}$ now arises for all modes $m_\varphi > 1$, as shown by ROSENBLUTH *et al.* [1962] and TAYLOR [1962]. A stabilizing effect by Coriolis' force still exists and is

represented by the unity term in the square root of $1 - m_\varphi$. However, this effect is now too small compared to the destabilizing contribution which is represented by m_φ.

(iii) $n \neq 0$, $m_\varphi \neq 0$ and $n \neq m_\varphi$. The sign of the expression inside the square root of (8.57) depends upon the magnitude and sign of rN'/Nm_φ^2, n^2/m_φ^2 and n. There are stable as well as unstable modes.

Thus, we have seen that the Coriolis force should have a stabilizing effect on the flute disturbance, but this effect is not sufficient to secure stability for all modes. To study this problem more in detail, we have to introduce the boundary conditions of the special configurations of interest. Such an analysis is out of the scope of the present elementary study of localized perturbations.

2.8. CONCLUSIONS

A sufficient condition for stability in the present problems is that the magnetic field bends convexly towards the main plasma body or that the gravitation and centrifugal forces are antiparallel to the density gradient. This has been shown in the detailed study of the plasma motion and is also required from energy considerations.

When the magnetic field lines bend concavely towards the plasma body or the gravitation and centrifugal forces point in the direction of decreasing density, stability may still be secured at certain conditions. The simple energy considerations given here provide a necessary condition for instability to occur. However, they are not always sufficient since it has also to be proved that the particles actually can perform the required displacements to lower the state of potential energy. This is, in fact, also what is taken into account by the rigorous theory based on the energy principle, where the equations of motion are included.

In this chapter a number of mechanisms have been demonstrated which sometimes reduce the growth rates of the flute instabilities for a certain wave number range, and sometimes even stabilize the systems considered for all possible wave numbers. These mechanisms arise from a weakening of the driving forces by a finite density gradient and the Hall effect, from finite Larmor radius effects, from magnetic compression effects, from a separation current produced by the Coriolis' force and from the effect of a sheared magnetic field. In addition to these intrinsic mechanisms there are also such effects as that produced by a conducting wall. Thus, a short-circuit across the

magnetic field at the ends of a magnetic bottle may remove the charges along the magnetic field lines and suppress the driving force of a flute instability (ROSENBLUTH and LONGMIRE [1957], POST *et al.* [1960]). Finally the unperturbed electric field may introduce finite Larmor radius effects which influence stability as suggested in the experiments with OGRA by BOGDANOV *et al.* [1962].

As far as experimental and observational tests of the theories on flute instabilities are concerned the situation is not fully clear, and the theoretically predicted conditions for the occurrence of instabilities have only been partly verified. When experiments are compared with the results of a localized perturbation analysis it is not sufficient to discuss the stability of the more or less pronounced plasma boundary. It is of equal importance to study whether the *interior* of the plasma body is stable against flute disturbances or not.

One interesting example in cosmical physics is given by the radiation belts discovered by VAN ALLEN [1959]. These belts consist of charged particles trapped in the magnetic mirror field of the earth, as sketched in Figure 7.5. To be judged from observations, they appear to be stable. One way to explain this is to assume that the space charges arising from a flute disturbance are removed along the magnetic field lines and are short-circuited across the ionosphere which should act somewhat like a conducting end plate (POST *et al.* [1960]).

Another possible explanation is suggested by the results of § 2.6 of the present chapter. Under the special conditions treated here it has been shown that, if the magnetic field gradient is at least half as steep as the density gradient, the system should become stable for all wave numbers in the directions across the magnetic field. For the Van Allen belts there are no material walls near the convex boundary regions of the plasma. It is therefore possible that the present condition for stability, $B/|\nabla B| < 2N/|\nabla N|$, can be fulfilled also in these outer regions, especially as the magnetic field falls off steeply in the radial direction.

In plasma experiments in the laboratory the situation is somewhat different as compared to cosmical conditions because the plasma has to be contained inside a vessel of finite size. Thus, the density of the plasma in the outer regions is likely to fall off more steeply than the confining magnetic field and the values of L_{cN} would approach zero at the plasma boundary. This applies to a large number of magnetic mirror devices and similar arrangements. Stability may still be secured, or the growth rates of flutes will at least be reduced, by means of the stabilizing mechanisms just discussed.

It should also be observed that the driving forces and the growth rates of a flute instability may become reduced under certain experimental conditions, even if L_{cN} approaches zero at a vessel wall or at an electrode surface. This is expected to occur in a rotating plasma which is bounded by electrode surfaces at which the fluid velocity approaches zero on account of viscous forces. The angular velocity of rotation is then no longer constant in space, but gives rise to an equivalent, non-uniform gravitation field, $g = \Omega^2 r$, which vanishes at the electrode surfaces. From the stability point of view this corresponds roughly to a case where N and g in § 2.4 simultaneously tend to zero at the boundary. Similarly, a reduction of the growth rate of a flute disturbance is expected to occur in a plasma the temperature of which approaches zero at the boundary, e.g. at a metal wall. This latter case corresponds roughly to a situation where P_i, P_e and N all tend to zero at the boundary in the example of § 2.6.

The exponential growth rate of a flute disturbance predicted by the present linearized theory for an unstable state suggests that particles escape across the magnetic field at a rate about equal to the thermal velocity. As an example, this velocity is of the order of 10^5 m/sec for a hydrogen plasma at 10^6 °K. Particles are then likely to be lost from the confining magnetic field of a laboratory device within a few microseconds. This time appears in any case to be some hundred times shorter than the observed confinement times in a number of plasma experiments. The reason for this discrepancy is not yet fully understood, but it seems likely that the stabilizing mechanisms described here could account at least for a strongly reduced growth rate. We should, of course, at the same time keep in mind that the present conclusions have been drawn from a linearized theory. Considerable modifications may become necessary when non-linear effects are taken into account.

3. Screw Shaped Disturbances

In the flute instability mechanisms analysed in the preceding paragraph a charge separation is produced by drift motions which are perpendicular to the magnetic field. However, a charge separation can equally well be generated by a differential motion between ions and electrons *along* the magnetic field lines. This produces, in its turn, a transverse electric field and an electric drift motion which drives the particles across the magnetic field.

A theoretical study of such a mechanism was first developed for a lowly ionized plasma by KADOMTSEV and NEDOSPASOV [1960]. The results of the analysis were found to be in good agreement with earlier observed instability

phenomena in the positive column (LEHNERT [1958b], HOH and LEHNERT [1960]). The instability mechanism is based on a screw-shaped density perturbation which extends along the magnetic field lines as shown in Figure 8.7. In this figure is assumed that a plasma column with a radial density gradient $dN/dr < 0$ is immersed in a homogeneous, axial magnetic field. In the initial state, which is electrically neutral, the density distributions are perturbed by a screw-shaped disturbance of equal magnitude for ions and elec-

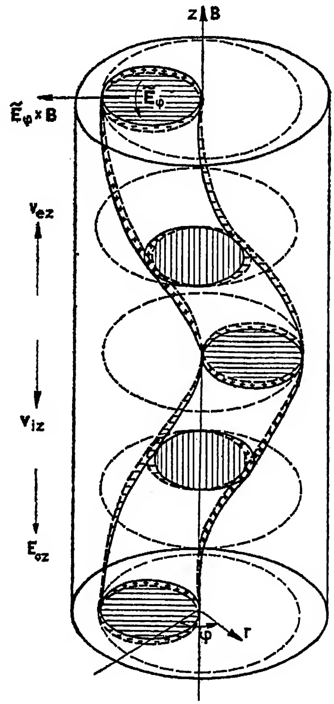


Fig. 8.7. Left-handed screw instability for $m_\varphi = 1$. The perturbed density distribution of ions is given by the screw-shaped body confined by full lines. The corresponding electron distribution is indicated by dashed lines. Cross sections are marked by shaded areas to indicate the positions of the screws relative to the z axis. Due to the axial motions v_{iz} and v_{ez} space charges will be generated in the regions where the perturbed distributions do not overlap (HOH and LEHNERT [1961]).

trons. At later times the latter move along the axis at different velocities, e.g. when an axial electric field E_{0z} is present. This will immediately produce a charge separation. Thus, an axial displacement of the "electron screw" in

the positive z direction of Figure 8.7 becomes equivalent to a rotation of the "electron screw" in the positive φ direction relative to the "ion screw". This, in its turn, generates the perturbed electric field $\tilde{\boldsymbol{E}}_\varphi$ and a drift $\tilde{\boldsymbol{E}}_\varphi \times \boldsymbol{B}/B^2$ outwards in the radial direction, provided that the screw has the given direction. The electric drift then moves the entire mass of plasma into positions of increasing eccentricity with respect to the axis. This makes the density deviations and the driving forces of the motion grow. There is always one sense of the screw in a system like that of Figure 8.7 for which a radial escape of plasma will take place.

The instability mechanism which has just been described resembles very much that of the flute instabilities of § 2. The only difference is that the longitudinal motion, and not a transverse drift, is responsible for the charge separation.

In a fully ionized plasma we should still expect such charge separation phenomena to occur as pictured in Figure 8.7. They are produced by a longitudinal convection of surfaces of constant density as given by the term $\boldsymbol{v}_\parallel \cdot \nabla n$ of equation (5.62). However, this is not the only effect which produces space charges on account of a longitudinal motion. As seen from the last term of (5.62) there also exist compression effects in the longitudinal direction which influence the rates of change of the ion and electron densities. In presence of a perturbed longitudinal electric field $\tilde{E}_z$ non-uniform accelerations of electrons and ions will arise along the magnetic field lines, provided that the Coulomb collision frequency is small enough. The space charge formation is then strongly influenced by longitudinal compression effects, and the simple situation of Figure 8.7 does no longer apply. It should therefore be stressed that even very small electric field components along the magnetic lines of force will create a situation of great complexity where all kinds of approximations have to be carefully examined. A detailed discussion of these problems in the case of a fully ionized plasma of finite resistivity is beyond the scope of this book.

CHAPTER 9

RELATIVISTIC EFFECTS

The inclusion of relativistic effects in the problems of this volume does not involve any principal difficulties. Here we shall summarize only some of the more important questions regarding the motion of individual particles. For the foundations of the special theory of relativity the reader is referred to textbooks on the subject such as those by EINSTEIN [1950], BERGMANN [1950], RICHTMYER and KENNARD [1947] and MØLLER [1952].

1. Relativity and the Electromagnetic Field

The theory of relativity is based on the fact that the velocity of light in free space is invariant to the state of motion of the frame in which it is observed. Since light can be regarded as an electromagnetic wave motion one should expect the laws of the electromagnetic field to be consistent with the requirements of relativity. This has, in fact, also been the case long before the questions about the theory of relativity even were raised.

1.1. THE LORENTZ TRANSFORMATION

A direct consequence of the invariance of the velocity c of light is the *Lorentz transformation.* Consider a frame C of reference where positions in space are indicated by the vector $\boldsymbol{\rho}$ and time is given by t. In another frame C′ which moves at the constant velocity $\boldsymbol{w}_0$ relative to the former the corresponding position vector is $\boldsymbol{\rho}'$. According to the requirements of special relativity a given light wave has to travel with the same velocity c in both coordinate systems. This is only possible if the time coordinate is changed simultaneously with the space coordinates when a transformation is made from one system to another. Consequently, we have to introduce the time t' in the moving frame, which differs from t. As a result we obtain the Lorentz transformation

$$\boldsymbol{\rho}' = \boldsymbol{\rho} + (\gamma - 1)(\hat{\boldsymbol{w}}_0 \cdot \boldsymbol{\rho})\hat{\boldsymbol{w}}_0 - \gamma \boldsymbol{w}_0 \cdot t, \tag{9.1}$$

$$t' = \gamma (t - \boldsymbol{w}_0 \cdot \boldsymbol{\rho}/c^2), \tag{9.2}$$

where

$$\gamma(w_0) = \frac{1}{(1 - w_0^2/c^2)^{\frac{1}{2}}} \tag{9.3}$$

and we have assumed the origins of both frames to coincide at time $t = 0$. Especially, if the x axis of the frames is orientated along $\boldsymbol{w}_0$ the transformation becomes

$$x' = \gamma(x - w_0 t), \quad y' = y, \quad z' = z, \tag{9.4}$$

$$t' = \gamma(t - w_0 x/c^2), \tag{9.5}$$

where $w_0 \equiv w_{0x}$ with the sign of w_{0x} included. This notation is used in all places where w_0 appears in this chapter.

From equations (9.1) and (9.2) we can deduce a relation between the velocities $\boldsymbol{w} = \mathrm{d}\boldsymbol{\rho}/\mathrm{d}t$ and $\boldsymbol{w}' = \mathrm{d}\boldsymbol{\rho}'/\mathrm{d}t'$ at which the position vector is observed to move in the two coordinate systems:

$$\boldsymbol{w}' = \frac{\mathrm{d}\boldsymbol{\rho}'}{\mathrm{d}t'} = \frac{\mathrm{d}\boldsymbol{\rho}'}{\mathrm{d}t}\frac{'\mathrm{d}t}{\mathrm{d}t'}$$

$$= \frac{\boldsymbol{w} + (\gamma - 1)(\hat{\boldsymbol{w}}_0 \cdot \boldsymbol{w})\hat{\boldsymbol{w}}_0 - \gamma \boldsymbol{w}_0}{\gamma(1 - \boldsymbol{w}_0 \cdot \boldsymbol{w}/c^2)}. \tag{9.6}$$

This equation reduces to the familiar law of addition of relativistic velocities if the x axis is chosen along $\boldsymbol{w}_0$.

According to MINKOWSKI [1915] we can regard space and time as four coordinates in a four-dimensional continuum. Consequently we introduce:

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = \mathrm{i}ct \tag{9.7}$$

and the Lorentz transformation (9.4) and (9.5) becomes

$$x_1' = x_1 \cos\eta - x_4 \sin\eta, \quad x_2' = x_2, \quad x_3' = x_3, \tag{9.8}$$

$$x_4' = x_1 \sin\eta + x_4 \cos\eta, \tag{9.9}$$

where

$$\mathrm{tg}\,\eta = -\mathrm{i}w_0/c, \quad \cos\eta = \gamma(w_0), \quad \sin\eta = -\mathrm{i}\gamma(w_0)w_0/c. \tag{9.10}$$

These results imply that the Lorentz transformation simply becomes a rotational transformation in a Cartesian $x_1x_2x_3x_4$ space, where η plays the role of an imaginary angle of rotation.

1.2. ELECTRIC CURRENTS AND CHARGES

The sources of the electromagnetic field are the electric charges and their motion. It is therefore natural to start the analysis with the questions how the electric current density and the charge density behave in a Lorentz transformation.

For this purpose we assume that an electric charge distribution of density σ moves with velocity $\boldsymbol{w}$ with respect to the frame C. The corresponding quantities are σ' and $\boldsymbol{w}'$ in the frame C′ which moves at velocity $\boldsymbol{w}_0$ with respect to C. Introduce a third frame C″ which follows the moving charge distribution, i.e. where the velocity is $\boldsymbol{w}'' = 0$ and the charge density becomes σ''. Consider a fixed number of charged particles which occupy a certain volume element. Denote the element in the frame C″ by $\mathrm{d}V''$. The electric charge inside the element is then $\sigma''\mathrm{d}V''$. From C″ the frames C and C′ are observed to move at velocities $-\boldsymbol{w}$ and $-\boldsymbol{w}'$. The same charged particles will then be seen to occupy the volumes $\mathrm{d}V$ and $\mathrm{d}V'$ in C and C′, respectively.

According to the Lorentz transformation (9.4) a contraction of an observed, moving volume takes place in the direction of relative motion. If we apply this to the transformation between the pairs of frames C″, C and C′, C the result becomes

$$\mathrm{d}V = \frac{\mathrm{d}V''}{\gamma(w)}, \quad \mathrm{d}V' = \frac{\mathrm{d}V''}{\gamma(w')}. \tag{9.11}$$

With the x axis along $\boldsymbol{w}_0$ equation (9.6) can easily be used to derive an expression for $(w')^2$ from which

$$\gamma(w') = \gamma(w) \cdot \gamma(w_0)(1 - w_x w_0/c^2). \tag{9.12}$$

Since we consider a volume element containing the same number of particles we must have $\sigma''\mathrm{d}V'' = \sigma\mathrm{d}V = \sigma'\mathrm{d}V'$ and combination of this condition with equations (9.11) and (9.12) yields

$$\sigma' = \gamma(w_0)(\sigma - w_0\sigma w_x/c^2). \tag{9.13}$$

If there are different kinds of particles the total current and charge densities become

$$\boldsymbol{j} = \sum_\nu \sigma_\nu \boldsymbol{w}_\nu, \quad \sigma = \sum_\nu \sigma_\nu \tag{9.14}$$

in the frame C and similar relations hold for the frame C′. For the total current and charge densities we then obtain from equations (9.6), (9.13) and (9.14)

$$j'_x = \gamma(w_0)(j_x - w_0\sigma), \quad j'_y = j_y, \quad j'_z = j_z, \tag{9.15}$$

$$\sigma' = \gamma(w_0)(\sigma - w_0 j_x/c^2). \tag{9.16}$$

Observe that the space charge density σ' depends upon the frame of reference in which it is being measured. This is not only so because of the Lorentz contraction factor γ in front of the right hand member of equation (9.16), but also because of the last term containing j_x in this member. The physical reason for this term to appear is that ions and electrons move with different velocities when a current j_x is flowing. This causes the Lorentz contractions to differ slightly for ions and electrons and produces a corresponding difference in their particle densities, i.e., a space charge $-\gamma(w_0) w_0 j_x/c^2$ arises. This effect is not only of importance to highly relativistic particles where $\gamma(w_0)$ differs noticeably from unity. As shown by FÄLTHAMMAR [1962] for an expanding stream of magnetized plasma the effect may even be of considerable importance to rather slow particles. This is so, because the slightest difference between very small Lorentz contractions of ions and electrons in a quasi-neutral plasma gives rise to space charges and appreciable electric fields, if the density of the plasma is high enough.

A comparison between equations (9.15) and (9.16) on one hand and equations (9.4) and (9.5) on the other shows that $\boldsymbol{j}$ and σ behave as space and time coordinates. The quantity $(\boldsymbol{j}, \mathrm{i}c\sigma)$ then transforms as coordinate differences in fourspace, i.e. it becomes a *four-vector*.

1.3. THE ELECTROMAGNETIC FIELD TENSOR

From the sources represented by $\boldsymbol{j}$ and σ we can calculate the magnetic vector potential $\boldsymbol{A}$ and the electric potential ϕ as indicated in Ch. 2, § 1.1. It is easily seen that the operators $\partial/\partial x_\nu$ transform as the components of a four-vector. Since the coordinates x_ν are given by equation (9.7) this implies that the operator $\nabla^2 - (1/c^2)\,\partial^2/\partial t^2$ becomes invariant in a Lorentz transformation. Consequently, equations (2.17) and (2.18) show that $\boldsymbol{A}$ and ϕ transform in the same way as $\boldsymbol{j}$ and σ, and the quantity

$$(A_\nu) = (\boldsymbol{A}, \mathrm{i}\phi/c) \tag{9.17}$$

will become a four-vector as well.

In the four-dimensional space given by (9.7) we can now form an electro-

magnetic field tensor by derivation of A_ν:

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} = -F_{\nu\mu}. \tag{9.18}$$

With this notation $\nu = 1, 2, 3$ corresponds to Maxwell's equation (2.8), and $\nu = 4$ to equation (2.10). The explicit expression for $F_{\mu\nu}$ follows from equations (9.18) and (9.7):

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & B_z & -B_y & -\frac{i}{c}E_x \\ -B_z & 0 & B_x & -\frac{i}{c}E_y \\ B_y & -B_x & 0 & -\frac{i}{c}E_z \\ \frac{i}{c}E_x & \frac{i}{c}E_y & \frac{i}{c}E_z & 0 \end{pmatrix}. \tag{9.19}$$

The transformation (9.8) and (9.9) of coordinates and coordinate differences can be written in the abbreviated form

$$x'_\mu = \sum_\nu a_{\mu\nu} x_\nu, \tag{9.20}$$

where $a_{\mu\nu}$ are the associated coefficients. From equations (9.17) and (9.18) it is then immediately seen that a corresponding transformation of the field tensor $F_{\mu\nu}$ is given by

$$F'_{jk} = \sum_\mu \sum_\nu a_{j\mu} a_{k\nu} F_{\mu\nu}. \tag{9.21}$$

The formulae which result from equations (9.21), (9.19), (9.20) and (9.10) for the transformation of the electromagnetic field can be written as

$$\boldsymbol{E}' = \gamma \boldsymbol{E} - (\gamma - 1)(\hat{\boldsymbol{w}}_0 \cdot \boldsymbol{E})\,\hat{\boldsymbol{w}}_0 + \gamma \boldsymbol{w}_0 \times \boldsymbol{B}, \tag{9.22}$$

$$\boldsymbol{B}' = \gamma \boldsymbol{B} - (\gamma - 1)(\hat{\boldsymbol{w}}_0 \cdot \boldsymbol{B})\,\hat{\boldsymbol{w}}_0 - \gamma \boldsymbol{w}_0 \times \boldsymbol{E}/c^2, \tag{9.23}$$

where $\gamma = \gamma(w_0)$.

2. Relativistic Equations of Motion

The laws of the electromagnetic field are consistent with the requirements of relativity. When we try to develop a relativistic theory on the mechanics

of matter, it is therefore convenient to start with the effects of the electromagnetic field on the motion of a charged particle. Following KLEIN [1950] we introduce an inertial frame of reference C in which the particle is observed to move at the velocity $\boldsymbol{w}(t_0) = \boldsymbol{w}_0$ at a particular instant $t = t_0$. Further introduce a frame C′ moving with respect to C at the constant velocity $\boldsymbol{w}_0$. Thus, the particle has the velocity zero in C′ at time $t = t_0$, but not at other times t. Without loss of generality we can assume the origins of both systems to coincide at time $t = t' = 0$ and put $t_0 = t_0' = 0$.

Since the velocity $\boldsymbol{w}'$ of the particle vanishes at $t' = 0$ we assume Newton's law to be valid in C′ at this time:

$$m \frac{\mathrm{d}\boldsymbol{w}'}{\mathrm{d}t'} = q\boldsymbol{E}', \quad t = t' = 0. \tag{9.24}$$

The question is now what form this relation adopts in terms of the field quantities measured in C. For this purpose we have to connect the acceleration $\mathrm{d}\boldsymbol{w}'/\mathrm{d}t'$ of the position vector in C′ with the corresponding acceleration $\mathrm{d}\boldsymbol{w}/\mathrm{d}t$ in C which moves at the constant velocity $-\boldsymbol{w}_0$ with respect to C′. A straightforward deduction from equations (9.6) and (9.2) yields

$$\left(\frac{\mathrm{d}\boldsymbol{w}'}{\mathrm{d}t'}\right)_{t'=0} = \left(\frac{\mathrm{d}\boldsymbol{w}'}{\mathrm{d}t}\,\frac{\mathrm{d}t}{\mathrm{d}t'}\right)_{t'=0} = \gamma^2\left[\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t} + (\gamma - 1)\left(\hat{\boldsymbol{w}}_0 \cdot \frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t}\right)\hat{\boldsymbol{w}}_0\right]_{t'=0}, \tag{9.25}$$

where the condition $\boldsymbol{w}'(0) = 0$ has been applied. The present deductions can be repeated for any time t_0 and any velocity $\boldsymbol{w}(t_0)$. Combination of equations (9.24), (9.25) and (9.22) therefore results in an equation of motion

$$m\gamma^2\left[\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t} + (\gamma - 1)\left(\hat{\boldsymbol{w}} \cdot \frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t}\right)\hat{\boldsymbol{w}}\right] = q\left[\gamma\boldsymbol{E} - (\gamma - 1)(\hat{\boldsymbol{w}} \cdot \boldsymbol{E})\hat{\boldsymbol{w}} + \gamma\boldsymbol{w} \times \boldsymbol{B}\right] \tag{9.26}$$

with $\gamma = \gamma(w)$. Scalar multiplication of this equation by $\hat{\boldsymbol{w}}$ gives

$$m\gamma^3\hat{\boldsymbol{w}} \cdot \frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t} = q\hat{\boldsymbol{w}} \cdot \boldsymbol{E}, \tag{9.27}$$

which is the relativistic correspondence to the energy equation (2.38).

Introduction of (9.27) and the expression $\mathrm{d}\gamma/\mathrm{d}t = (\gamma^3/c^2)\boldsymbol{w} \cdot \mathrm{d}\boldsymbol{w}/\mathrm{d}t$ into (9.26) gives after rearrangement of the resulting terms

$$m \frac{\mathrm{d}}{\mathrm{d}t}(\gamma \boldsymbol{w}) = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{P}) = q\,(\boldsymbol{E} + \boldsymbol{w} \times \boldsymbol{B}) \tag{9.28}$$

which is the relativistic equation of motion corresponding to (2.36). Here $\boldsymbol{P} = m\gamma\boldsymbol{w}$ represents the corresponding momentum vector. The energy equation (9.27) can also be rewritten in the form

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t} = q\boldsymbol{w} \cdot \boldsymbol{E}, \quad \mathscr{E} = \gamma mc^2, \tag{9.29}$$

where $\mathscr{E}$ is defined as the relativistic energy.

Now return to the four-dimensional representation of §§ 1.2 and 1.3. Introduce the *proper time* interval

$$\mathrm{d}\tau = [(\mathrm{d}t)^2 - (\mathrm{d}\rho)^2/c^2]^{\frac{1}{2}} = \mathrm{d}t/\gamma. \tag{9.30}$$

This quantity can also be considered as a measure of the arc length along the orbit of the position vector in $x_1x_2x_3x_4$ space because

$$\sum_{\nu} (\mathrm{d}x_\nu)^2 = -\,c^2(\mathrm{d}\tau)^2. \tag{9.31}$$

It therefore becomes invariant to changes of the coordinate system. Consequently, a four-velocity of the position vector in this space can be defined and becomes

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau} = \left(\frac{\mathrm{d}\rho}{\mathrm{d}t}\,\frac{\mathrm{d}t}{\mathrm{d}\tau},\ \mathrm{i}c\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\right) = (\gamma\boldsymbol{w}, \mathrm{i}c\gamma). \tag{9.32}$$

A corresponding four-dimensional momentum vector $(\boldsymbol{P}, \mathrm{i}\mathscr{E}/c)$ can also be introduced. By the aid of equations (9.32) and (9.19) relations (9.28) and (9.29) for the conservation of momentum and energy can now be assembled into one expression:

$$m\frac{\mathrm{d}^2x_\nu}{\mathrm{d}\tau^2} = q\sum_{\nu} F_{\mu\nu}\frac{\mathrm{d}x_\mu}{\mathrm{d}\tau}, \quad \nu = (1, 2, 3, 4). \tag{9.33}$$

Observe that this system of four equations is overdetermined. The relation for $\nu = 4$ expresses conservation of energy which follows from the first three equations, where $\nu = 1, 2, 3$ according to the notation of equation (9.7).

3. Special Applications to the Equation of Motion

Among the exact solutions of the equations of motion only two simple examples will be given here which are closely connected with the problems of Ch. 2, §§ 2 and 4.

3.1. MAGNETOSTATIC FIELD

When the electric field is absent and the particle moves only in a magnetostatic field the right hand member of equation (9.29) vanishes and the energy $\mathscr{E}$ as well as the modulus w of the velocity and γ become constants of the motion. The only difference between the non-relativistic and relativistic cases is then that the particle mass m should be replaced by the "relativistic mass" γm.

All results which are derived from the exact equation of motion in a magnetostatic field are then also valid in the relativistic case when this replacement is made. This is so for the problems of Chapter 2 in §§ 4.2 and 4.3 where the forbidden zones and the particle motion have been studied in the fields from a monopole, a dipole, a line current and in a hyperbolic field.

Since we have seen that a relativistic particle moves in exactly the same way as a non-relativistic particle of mass γm the results of the perturbation theory can also be directly translated to relativistic particles. Thus, when the velocity of gyration is $\boldsymbol{W}$ as seen from the laboratory frame, the radius of gyration should become $\gamma m W/|q|B$, and the equivalent magnetic moment becomes $\gamma m W^2/2B$.

When an electric field is present, however, γ and w are no longer constant and the perturbation theory has to be re-examined as outlined in § 4 of this Chapter.

3.2. HOMOGENEOUS MAGNETOSTATIC AND ELECTROSTATIC FIELDS

We shall now investigate the relativistic analogy to the problem of homogeneous magnetostatic and electrostatic fields treated in Ch. 2, § 4.1. When the field tensor $F_{\mu\nu}$ of (9.33) is constant we can follow HELLWIG [1955] and VANDERVOORT [1960] and study solutions of the form exp (const. τ). The roots of the characteristic equation are obtained from expression (9.19) which is inserted into (9.33). The general solution is of the form

$$x_\nu = \xi_\nu\, a \cos \omega\tau - \eta_\nu a \sin \omega\tau + \alpha_\nu b \cosh \lambda\tau - \beta_\nu b \sinh \lambda\tau + X_{\nu 0}, \quad (9.34)$$

where

$$\omega, \lambda = \frac{q}{m}\left\{\pm \tfrac{1}{2}\left(B^2 - \frac{E^2}{c^2}\right) + \tfrac{1}{2}\left[\left(B^2 - \frac{E^2}{c^2}\right)^2 + 4\frac{(\boldsymbol{E}\cdot\boldsymbol{B})^2}{c^2}\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}. \tag{9.35}$$

Here ω refers to the plus sign and ξ_ν, η_ν, α_ν, β_ν, a, b, and $X_{\nu 0}$ are constants. In the discussion which follows, we shall see that ξ_ν, η_ν, α_ν and β_ν can be chosen as orthogonal unit vectors.

The quantity a in equation (9.34) is associated with the part of the four-vector x_ν which "gyrates" at the frequency ω in $x_1x_2x_3x_4$ space. Thus, it can be regarded as a generalization of the Larmor radius (2.81) to the motion in four-dimensional space. The particle will then gyrate around the magnetic field lines at a radial distance which is obtained from the projection of the four-dimensional orbit into three-dimensional space. In the non-relativistic limit the quantity a of (9.34) approaches the value given by (2.81).

By insertion of the solution (9.34) into (9.33) we shall now verify that ξ_ν, η_ν, α_ν and β_ν can be chosen as orthogonal unit vectors. When terms with the same functional dependence upon τ are equated the result becomes

$$\omega\xi_\nu = \frac{q}{m}\sum_\mu F_{\nu\mu}\eta_\mu, \quad \omega\eta_\nu = -\frac{q}{m}\sum_\mu F_{\nu\mu}\xi_\mu \tag{9.36}$$

and

$$\lambda\alpha_\nu = -\frac{q}{m}\sum_\mu F_{\nu\mu}\beta_\mu, \quad \lambda\beta_\nu = -\frac{q}{m}\sum_\mu F_{\nu\mu}\alpha_\mu. \tag{9.37}$$

From equation (9.36) we have

$$\omega\sum_\nu \xi_\nu\eta_\nu = \frac{q}{m}\sum_\nu\sum_\mu F_{\nu\mu}\eta_\mu\eta_\nu = 0 \tag{9.38}$$

due to the antisymmetry of $F_{\nu\mu}$ expressed by (9.18). A result analogous to (9.38) holds for α_ν, β_ν. From (9.36) and (9.37) we further obtain

$$\omega\sum_\nu \xi_\nu\alpha_\nu = \frac{q}{m}\sum_\nu\sum_\mu F_{\nu\mu}\eta_\mu\alpha_\nu = -\lambda\sum_\nu \beta_\nu\eta_\nu, \tag{9.39}$$

$$\omega\sum_\nu \eta_\nu\beta_\nu = \lambda\sum_\nu \alpha_\nu\xi_\nu, \tag{9.40}$$

and

$$\omega\sum_\nu \xi_\nu\alpha_\nu = -\frac{\lambda^2}{\omega}\sum_\nu \alpha_\nu\xi_\nu = \frac{\lambda^3}{\omega^2}\sum_\nu \beta_\nu\eta_\nu, \tag{9.41}$$

which can only be satisfied if

$$\sum_\nu \xi_\nu \alpha_\nu = \sum_\nu \beta_\nu \eta_\nu = 0. \tag{9.42}$$

Finally, equation (9.36) yields

$$\omega \sum_\nu \xi_\nu \xi_\nu = \frac{q}{m} \sum_\nu \sum_\mu F_{\nu\mu} \eta_\mu \xi_\nu = \omega \sum_\nu \eta_\nu \eta_\nu \tag{9.43}$$

and a similar expression is obtained for α_ν and β_ν from equation (9.37). The results (9.38) and (9.42) show that the vectors ξ_ν, η_ν, α_ν, and β_ν are all orthogonal. Since a and b have not yet been specified we can let these quantities represent the amplitudes of the solution and put the vectors ξ_ν, η_ν, α_ν and β_ν equal to unit vectors in accordance with equation (9.43). Situations where one of the quantities ω and λ vanishes must be treated separately since it is not obvious that the present analysis then becomes applicable.

We shall close this discussion with two special cases:

(i) When the electric field is perpendicular to the magnetic field $\boldsymbol{E} \cdot \boldsymbol{B} = 0$, and λ vanishes. If $E^2 < c^2B^2$ at the same time, ω becomes real and different from zero. The determinant of $F_{\mu\nu}$ vanishes. It is then easily seen that the solution of (9.33) becomes

$$x_\nu = \xi_\nu\, a \cos \omega\tau - \eta_\nu\, a \sin \omega\tau + U_\nu \tau, \tag{9.44}$$

where the constant $X_{\nu 0}$ has been dropped. According to (9.33) the four-velocity U_ν should satisfy the condition

$$\sum_\nu F_{\mu\nu} U_\nu = 0 \tag{9.45}$$

which is only possible for non-trivial solutions of $U_\nu \neq 0$ if the determinant of $F_{\mu\nu}$ vanishes.

In analogy with (9.32) introduce the notation

$$(U_\nu) = (\boldsymbol{U}, \mathrm{i}cU_t) \tag{9.46}$$

where $\boldsymbol{U}$ represents the "space part" and $U_t = \mathrm{d}t/\mathrm{d}\tau = \gamma$ the "time part" of U_ν. Equations (9.45) and (9.19) then yield

$$\boldsymbol{U} \times \boldsymbol{B} + \boldsymbol{E} U_t = 0 \tag{9.47}$$

and

$$\boldsymbol{E} \cdot \boldsymbol{U} = 0. \tag{9.48}$$

In the non-relativistic limit U_t approaches unity and $\boldsymbol{U}$ tends to $\boldsymbol{u}_\perp$. Equations (9.47) and (9.48) are therefore analogous to (3.26) and (2.38) for the motion in homogeneous fields. The terms containing $\boldsymbol{a}$ in (9.44) are the relativistic

analogy to the gyration which is expressed by (2.80) and which is superimposed on the drift motion of the guiding centre. The latter is represented by U_ν. In particular, U_t will be associated with the energy of the guiding centre motion.

(ii) Another extreme case is that where the magnetic and electric fields are parallel. With the z axis along these fields (9.33) divides into the two sets

$$\frac{d^2x}{d\tau^2} = \omega_g \frac{dy}{d\tau}, \qquad \frac{d^2y}{d\tau^2} = -\omega_g \frac{dx}{d\tau} \tag{9.49}$$

and

$$\frac{d^2z}{d\tau^2} = \frac{qE_z}{m} \frac{dt}{d\tau}, \qquad \frac{d^2t}{d\tau^2} = \frac{qE_z}{mc^2} \frac{dz}{d\tau}, \tag{9.50}$$

where ω_g is the non-relativistic frequency of gyration.

The first set (9.49) gives a solution for x and y of the form represented by the first two terms of the right hand member of (9.34) with $\omega = \omega_g$. In the non-relativistic limit this solution tends to the gyration described by (2.80).

The second set (9.50) corresponds to the third and fourth terms of the right hand member of the solution (9.34). It yields non-periodic solutions for z and t with $\lambda = qE_z/mc$. In the non-relativistic limit the solution of the first of equations (9.50) is simply a uniformly accelerated motion of the guiding centre along the magnetic field.

* 4. Perturbation Theory

In analogy with the orbit theory of Chapter 3 the motion of the particle in $x_1x_2x_3x_4$ space can now be divided into a rapidly fluctuating periodic motion superimposed on a slow drift. This problem was first considered by HELLWIG [1955] and the theory was further developed by VANDERVOORT [1960] along lines which will be summarized in this paragraph.

A relativistic formulation of the equations of motion makes possible not only a treatment of high-energy particles. In a non-relativistic theory we have restricted the discussion to strong magnetic fields with a crossed, or nearly crossed electric field, as expressed by equation (3.34). Further, the electric field drift $\boldsymbol{u}_E$ cannot be allowed to approach the velocity of light in such a theory. In a relativistic theory these restrictions are largely removed.

4.1. STARTING POINTS AND BASIC RELATIONS

When the variations of the electromagnetic field in space and time are slow we expect the particle motion to deviate only little from the case of a

uniform field. Consider a field which is uniform for $\tau \leqslant \tau_0$ and which becomes slightly non-uniform for $\tau > \tau_0$. Imagine that, as in the problems of § 3.2, we can separate the particle motion into a rapidly fluctuating gyration and a drift of a guiding centre. In analogy with equation (9.34) we therefore assume the position vector of the particle to be given by

$$x_\nu = \xi_\nu \, a \cos \omega\tau - \eta_\nu \, a \sin \omega\tau + \tilde{x}_\nu + X_\nu, \tag{9.51}$$

where the two first terms of the right hand member describe the gyration. Here ξ_ν and η_ν should satisfy equations (9.36) and (9.43) while ω is given by equation (9.35). The quantity X_ν represents the position vector of all such contributions which do not fluctuate rapidly and periodically in time, i.e. it is the relativistic counterpart to the position vector $\boldsymbol{C}$ of the guiding centre in equation (3.6). Finally, we have to add a small, periodic contribution $\tilde{x}_\nu$ to the motion. This contribution is due to the non-uniformity of the field. In the non-relativistic limit its correspondence is due to the fact that the velocity of gyration becomes slightly inclined to the curved field lines of an inhomogeneous magnetic field, at least during parts of a Larmor period. We shall specify $\tilde{x}_\nu$ later.

The coming analysis turns out to become simplified if a change of variables is made:

$$\zeta_a = \frac{a \cos \omega\tau - ia \sin \omega\tau}{\sqrt{2}}, \quad \zeta_a{}^* = \frac{a \cos \omega\tau + ia \sin \omega\tau}{\sqrt{2}}. \tag{9.52}$$

Further introduce the vectors

$$\sigma_\nu = \frac{\xi_\nu + i\eta_\nu}{\sqrt{2}}, \quad \delta_\nu = \frac{\xi_\nu - i\eta_\nu}{\sqrt{2}} \tag{9.53}$$

which obey the relations

$$\sum_\nu \sigma_\nu \sigma_\nu = \sum_\nu \delta_\nu \delta_\nu = 0, \quad \sum_\nu \sigma_\nu \delta_\nu = 1 \tag{9.54}$$

according to equations (9.38) and (9.43).

The position of the particle is now given by

$$x_\nu = \delta_\nu \zeta_a + \sigma_\nu \zeta_a{}^* + \tilde{x}_\nu + X_\nu. \tag{9.55}$$

In this notation ζ_a and $\zeta_a{}^*$ describe the principal motion which is essentially a gyration around the field lines. The remaining part of the periodic motion which does not lie in the $\sigma_\nu \delta_\nu$ plane, i.e. not in the plane of ξ_ν and η_ν, is

represented by $\tilde{x}_\nu$. Thus we require

$$\sum_\nu \sigma_\nu \tilde{x}_\nu = \sum_\nu \delta_\nu \tilde{x}_\nu = 0. \tag{9.56}$$

Because the motions represented by $\tilde{x}_\nu$ do not occur in a uniform field, we shall assume that $|\tilde{x}_\nu|$ is small compared to $|\zeta_a|$; this can also be verified by the perturbation theory to be developed (VANDERVOORT [1960]). Further, in the limit of a uniform field we expect X_ν to tend to the fundamental solution represented by the last three terms of equation (9.34) when $\lambda \neq 0$, and to the term $U_\nu \tau$ of equation (9.44) when $\lambda = 0$. In either case, the component of the guiding centre velocity $U_\nu = \mathrm{d}X_\nu/\mathrm{d}\tau$ in the $\sigma_\nu \delta_\nu$ plane will be small.

We shall specify the assumption of slow field variations by the conditions

$$a \left| \frac{\partial F_{\mu\nu}}{\partial x_\sigma} \right| / \left| F_{\mu\nu} \right| \ll 1 \tag{9.57}$$

and

$$\frac{1}{\omega} \left| \frac{\partial F_{\mu\nu}}{\partial x_\sigma} \right| \cdot |U_\delta| / |F_{\mu\nu}| \ll 1 \tag{9.58}$$

in analogy with conditions (3.1) and (3.2). Suppose that L_c is the characteristic length of an interval in $x_1x_2x_3x_4$ space in which the change of $F_{\mu\nu}$ is comparable to $F_{\mu\nu}$ and that $U_{\nu c}$ is a typical component of the four-velocity. Conditions (9.57) and (9.58) then imply that

$$\varepsilon = |U_{\nu c}/\omega L_c| \ll 1, \tag{9.59}$$

where ε can be considered as a smallness parameter in analogy with equation (3.3).

Inequalities of this kind apply to any function $\chi(X_\nu)$ which depends on space and time through the components of $F_{\mu\nu}$. Thus, $|\partial\chi/\partial x_\nu|/|\chi/a|$ and $|\mathrm{d}\chi/\mathrm{d}\tau|/|\omega\chi|$ are of order ε. However, for

$$\frac{\mathrm{d}^2\chi}{\mathrm{d}\tau^2} = \sum_\mu \sum_\nu \frac{\partial^2 \chi}{\partial x_\mu \partial x_\nu} U_\mu U_\nu + \sum_\nu \frac{\partial \chi}{\partial x_\nu} \cdot \frac{\mathrm{d}U_\nu}{\mathrm{d}\tau} \tag{9.60}$$

we observe that the first term of the right hand member is of second order in ε, but the second term is not always so. When $\boldsymbol{E} \cdot \boldsymbol{B} \neq 0$ the guiding centre may experience large accelerations and $\mathrm{d}U_\nu/\mathrm{d}\tau$ may become comparable to ωU_ν. Therefore, we must treat $\mathrm{d}^2\chi/\mathrm{d}\tau^2$ as a quantity of order ε compared to $\omega^2\chi$, if $\boldsymbol{E} \cdot \boldsymbol{B} \neq 0$.

4.2. THE EQUATION OF MOTION OF THE GUIDING CENTRE

The particle orbit to be determined is now characterized by the eight unknown quantities included in equation (9.55). They are the four components X_ν of the guiding centre, the two quantities ζ_a and $\zeta_a{}^*$ describing the gyration, and the two periodic components $\tilde{x}_\nu$ which are perpendicular to the $\sigma_\nu \delta_\nu$ plane. The equation of motion (9.33) provides only four relations, but an additional set of four conditions will arise from the assumption that the motion can be split into an oscillating motion superimposed on a drift of the guiding centre.

A deduction of the equation of motion of the guiding centre can now be performed in a way analogous to that presented in Chapter 3 for the non-relativistic case. We shall not give the details of the theory. It can be summarized as follows:

(i) When conditions (9.57) and (9.58) are satisfied a Taylor expansion of the field tensor $F_{\mu\nu}$ can be carried out.

(ii) The resulting expression is inserted into the equation of motion (9.33).

(iii) The problem is solved by an iteration scheme. In the first approximation all second order terms are neglected and expressions for X_ν, ζ_a and $\zeta_a{}^*$ are determined. These expressions can be inserted into the equation of motion to obtain higher order approximations.

(iv) Thus, in equation (9.33) we concentrate on terms which are linear in ζ_a. By this we have separated an equation of motion for the gyration from the total equation of motion for x_ν.

(v) From the latter equation we also separate the non-oscillatory terms. These lead to an equation of motion for the velocity U_ν of the guiding centre. In the first approximation it becomes

$$m \frac{\mathrm{d}U_\nu}{\mathrm{d}\tau} = q \sum_\mu F_{\mu\nu} U_\mu - m\zeta_{a0}^2 \omega_0 \frac{\partial \omega}{\partial x_\nu}, \tag{9.61}$$

where ω_0 and ζ_{a0} are values at the starting point. The quantity $q\omega_0\zeta_{a0}^2$ is the relativistic correspondence to the equivalent magnetic moment. Equation (9.61) is analogous to (3.16). Observe that (9.61) involves only the variable X_ν. In the first approximation the guiding centre motion therefore becomes

independent of the details of the gyration. That (9.61) separates from the equation of motion of the gyration is just what makes possible the iteration scheme mentioned under (iii).

(vi) Starting with the first order solution of X_ν from (9.61) we can now determine the gyration from its corresponding equation of motion. Combination of the first order solutions for X_ν and ζ_a, ζ_a^* with the equation of motion for x_ν then yields an equation of motion for $\tilde{x}_\nu$.

(vii) The solution of the particle orbit can then be carried out to the second order approximation. For details reference is made to the original work by Vandervoort [1960].

4.3. THE EQUIVALENT MAGNETIC MOMENT

The quantity

$$M = \frac{q}{\omega}\left|\frac{\mathrm{d}\zeta_a}{\mathrm{d}\tau}\right|^2 \tag{9.62}$$

can be regarded as a generalization of the equivalent magnetic moment to a relativistic case. This is seen from equation (9.52) which yields

$$M = q|\zeta_a|^2\,\omega = \tfrac{1}{2}qa^2\omega \tag{9.63}$$

and where ζ_a describes the gyration of the particle. In the non-relativistic limit a tends to expression (2.81), ω approaches qB/m, and equation (9.63) reduces to equations (4.65) and (2.83).

The adiabatic invariance of M can be examined by means of the methods by Chandrasekhar [1958] which are outlined in Ch. 4, § 2.1. Consequently, we consider a situation where the particle is in a non-uniform field during a proper time interval $\tau_0 < \tau < \tau_f$ and in a uniform field for $\tau \leqslant \tau_0$ and $\tau \geqslant \tau_f$. In analogy with (4.67) we suppose that the velocity of the particle and the variation of the field are such that

$$\left|\omega \bigg/ \frac{\mathrm{d}\omega}{\mathrm{d}\tau}\right|_{\min} = \tau_\omega. \tag{9.64}$$

We then pass to the limit of infinitely slow field variations by letting $\tau_0 \to -\infty$, $\tau_f \to +\infty$, and $\tau_\omega \to \infty$. If in this limit $\Delta M = M(\tau \to +\infty) - M(\tau \to -\infty) \to 0$, we say that M is an adiabatic invariant.

By using the second order expression for $\mathrm{d}\zeta_a/\mathrm{d}\tau$ deduced from the equation of motion Vandervoort [1960] finds that ΔM tends to zero at least more

rapidly than the second power of ε. Thus, M is at least an adiabatic invariant in second order.

4.4. MOTION OF THE GUIDING CENTRE IN NEARLY CROSSED ELECTRIC AND MAGNETIC FIELDS

In the case that $\boldsymbol{E}$ is nearly perpendicular to $\boldsymbol{B}$ the component $\boldsymbol{E}_{\parallel}$ is small. Besides ε we can then introduce an additional smallness parameter which according to (9.35) becomes

$$\frac{\lambda}{\omega} \approx \frac{\boldsymbol{E}\cdot\boldsymbol{B}/c}{B^2 - E_{\perp}^2/c^2} \ll 1. \tag{9.65}$$

We restrict ourselves to the case where $E^2 < c^2B^2$, so that ω remains finite when $\boldsymbol{E}_{\parallel}$ and λ tend to zero. We shall not distinguish between ε and λ/ω when terms of different orders are separated, but shall bear in mind that the assumptions of the smallness of ε and λ/ω represent quite different approximations. Consequently, we regard $E_{\parallel}$ and λ as quantities of first order.

Resolve $F_{\mu\nu}$ into one part $F_{\mu\nu}^{(0)}$ constructed from $\boldsymbol{B}$ and $\boldsymbol{E}_{\perp}$, and one part $F_{\mu\nu}^{(1)}$ constructed from $\boldsymbol{E}_{\parallel}$. Also resolve the four-velocity (9.46) of the guiding centre into two parts of zero and first order:

$$U_\nu = U_\nu^{(0)} + U_\nu^{(1)}. \tag{9.66}$$

The non-zero eigenvalue of $F_{\mu\nu}^{(0)}$ is

$$\omega^{(0)} = (q/m)(B^2 - E_{\perp}^2/c^2)^{\frac{1}{2}}. \tag{9.67}$$

With these starting points and with U_ν expressed by (9.46) the equation (9.61) of motion yields

$$\sum_{\mu} F_{\nu\mu}^{(0)}\cdot U_{\mu}^{(0)} = 0 \tag{9.68}$$

and

$$m\frac{\mathrm{d}U_\nu^{(0)}}{\mathrm{d}\tau} = q\sum_{\mu}(F_{\nu\mu}^{(0)}U_{\mu}^{(1)} + F_{\nu\mu}^{(1)}U_{\mu}^{(0)}) - m\zeta_{a0}^2\omega_0\frac{\partial\omega}{\partial x_\nu}. \tag{9.69}$$

Up to this point we have mainly considered the motion in a four-dimensional representation. For practical purposes, one may instead be interested in explicit expressions of the space and time dependence. From the components of (9.69) in three-dimensional space follows that

$$m\frac{\mathrm{d}\boldsymbol{U}^{(0)}}{\mathrm{d}\tau} = q(\boldsymbol{U}^{(1)}\times\boldsymbol{B} + \boldsymbol{E}_{\perp}U_t^{(1)} + c\boldsymbol{E}_{\parallel}U_t^{(0)}) - m\zeta_{a0}^2\omega_0\nabla\omega \tag{9.70}$$

and the time coordinate yields

$$mc^2 \frac{dU_t^{(0)}}{d\tau} = q(\boldsymbol{E}_\perp \cdot \boldsymbol{U}^{(1)} + \boldsymbol{E}_\parallel \cdot \boldsymbol{U}^{(0)}) + m\zeta_{a0}^2\omega_0 \frac{\partial\omega}{\partial t}. \tag{9.71}$$

Equation (9.70) expresses conservation of momentum and equation (9.71) conservation of energy.

Equation (9.68) can be written as

$$\boldsymbol{U}^{(0)} \times \boldsymbol{B} + \boldsymbol{E}_\perp U_t^{(0)} = 0, \quad \boldsymbol{E}_\perp \cdot \boldsymbol{U}^{(0)} = 0 \tag{9.72}$$

and $U_\nu^{(0)}$ therefore has the form

$$U_\nu{}^{(0)} = (\hat{\boldsymbol{B}} U_\parallel^{(0)} + \boldsymbol{u}_E U_t^{(0)},\ \mathrm{i}c\, U_t{}^{(0)}) \tag{9.73}$$

with $\boldsymbol{u}_E = \boldsymbol{E} \times \boldsymbol{B}/B^2$. Here $U_\parallel{}^{(0)}$ represents the component of the four-velocity in the direction of the magnetic field and $U_t{}^{(0)}$ corresponds to the slowly varying part of the energy of the particle.

The equation of motion for $U_\nu{}^{(1)}$ consists of the components of (9.70) which are transverse to the magnetic field.

We now express the four-velocities of equations (9.70) and (9.71) in their space and time components in accordance with equations (9.46) and (9.32). We further allow $\boldsymbol{E}_\perp$ and the time dependence to be of zero order in ε, but assume $\boldsymbol{E}_\parallel$ to be of first order as stated before. According to NORTHROP [1961] the result corresponding to equations (9.70) and (9.71) then becomes:

$$\begin{aligned} \boldsymbol{u}_\perp = \Bigg\{ & q\left(1 - \frac{E_\perp^2}{c^2B^2}\right)\boldsymbol{E} - \frac{M}{\gamma}\nabla\left[B\left(1 - \frac{E_\perp^2}{c^2B^2}\right)^{\frac{1}{2}}\right] \\ & - m\gamma\left(u_\parallel \frac{d\hat{\boldsymbol{B}}}{dt} + \frac{d\boldsymbol{u}_E}{dt}\right) - q\left(\frac{u_\parallel E_\parallel}{c^2}\right)\boldsymbol{u}_E \\ & - \frac{M}{\gamma c^2}\boldsymbol{u}_E \frac{\partial}{\partial t}\left[B\left(1 - \frac{E_\perp^2}{c^2B^2}\right)^{\frac{1}{2}}\right]\Bigg\} \times \frac{\boldsymbol{B}}{qB^2(1 - E_\perp^2/c^2B^2)} + \mathrm{O}(\varepsilon^2), \end{aligned} \tag{9.74}$$

$$m\frac{d(\gamma u_\parallel)}{dt} = m\gamma \boldsymbol{u}_E \cdot \frac{d\hat{\boldsymbol{B}}}{dt} + qE_\parallel - \frac{M}{\gamma}\frac{\partial}{\partial s}[B(1 - E_\perp^2/c^2B^2)^{\frac{1}{2}}] \tag{9.75}$$

and

$$\frac{d}{dt}(mc^2\gamma) = q\boldsymbol{E}\cdot\boldsymbol{u} + \frac{M}{\gamma}\frac{\partial}{\partial t}[B(1 - E_\perp^2/c^2B^2)^{\frac{1}{2}}]. \tag{9.76}$$

In equations (9.74)–(9.76)

$$M = \frac{(P_\perp{}^*)^2}{2mB(1 - E_\perp^2/c^2B^2)^{\frac{1}{2}}}, \tag{9.77}$$

where $P_\perp{}^*$ is equal to the perpendicular relativistic momentum of the particle as observed from a frame moving at the electric drift velocity $\boldsymbol{u}_E$. The quantity M is a relativistic counterpart to the equivalent magnetic moment of equation (2.83). It can be shown to become proportional to the flux through the circle of gyration, as observed in a frame where the electric and magnetic fields are parallel (VANDERVOORT [1960]).

The results (9.74) and (9.75) are analogous to equations (3.18) and (3.17), and equation (9.76) expresses the balance of energy. The first term within the large bracket of (9.74) corresponds to the electric field drift (3.26). The second term is analogous to the first part of the magnetic gradient drift (3.24). Further, the third term has one contribution corresponding to the magnetic gradient drift (3.24) from the curvature of the field lines, and one part which is analogous to the polarization drift (3.27). Finally, the two last terms within the large bracket represent purely relativistic effects with no counterpart in the theory of Chapter 3.

The present result has been deduced for a general situation where $E_\perp$ is of zero order and the transverse electric field drift becomes very large. If $E_\perp$ is only of first order and $E_\perp^2 \ll c^2B^2$ the quantity $E_\perp^2/c^2B^2$ in equations (9.74)—(9.77) can be dropped and terms containing $\boldsymbol{u}_E$ are negligible. The solution then turns into a form earlier suggested by NORTHROP and TELLER [1960]. The values of the momentum, the radius of gyration and the equivalent magnetic moment are compared to those of the non-relativistic case in Table 9.1.

TABLE 9.1.

Comparison between non-relativistic and relativistic quantities when the electric field drift and the characteristic time variations are of first order in ϵ.

Quantity	Non-relativistic case	Relativistic case
Velocity of gyration as seen from laboratory frame	$\boldsymbol{W}$	$\boldsymbol{W}$
Perpendicular momentum due to Larmor motion	$m\boldsymbol{W}$	$\gamma m\boldsymbol{W}$
Larmor radius	$mW/\lvert q\rvert B$	$\gamma mW/\lvert q\rvert B$
Equivalent magnetic moment	$(mW)^2/2mB$	$(\gamma mW)^2/2mB$

Equation (9.74) tends to equation (3.18) in the non-relativistic limit. When the electric field vanishes and $\boldsymbol{F} = 0$ the only difference between equations (9.74) and (3.18) lies in the factor γ. This is expected from the discussion in § 3.1 according to which the orbit of a relativistic particle can be obtained from the non-relativistic equation of motion of a particle of the same velocity and the same "relativistic mass", γm.

* RADIATION

This chapter deals with the electromagnetic radiation emitted from charged particles. An extensive survey of the problem has earlier been given by HEITLER [1954] and reviews of parts of the subject are also due to STRATTON [1941], RICHTMYER and KENNARD [1947], and SCHIFF [1949]. Summaries of radiation problems of special interest in plasma physics have been given by CLAUSER [1960], ROSE and CLARK [1961] and BEKEFI and BROWN [1961].

The present discussion on radiation phenomena will be based on classical theory and excludes quantum mechanical effects. In our study of the radiation from an ionized gas we shall also assume ions and electrons to have thermal velocity distributions. This implies that non-thermal radiation due to instabilities, waves and other co-operative phenomena will be excluded.

1. The Radiation Problem in Plasma Physics

Radiation plays an important rôle both in astrophysics and in the research on high temperature plasmas in the laboratory. The electromagnetic energy emitted from a charged particle increases steeply with its acceleration. In a fully ionized gas of high temperature the density of radiation therefore becomes considerable.

Especially in the research on plasmas at thermonuclear temperatures radiation constitutes a severe loss, the problems of which have not yet been settled. If the Stefan-Boltzmann radiation law were applicable to a plasma under thermonuclear conditions, the energy of a typical thermonuclear system would be drained in times of the order of 10^{-17} seconds. Fortunately, when we consider frequencies well above the plasma frequency and the gyro frequency the mechanisms for excitation and radiation become extremely weak, and the plasma is optically very thin. In fact, one would not expect frequencies above a level corresponding to 10^{-3} eV to be in radiative equilibrium. However, even this small region of the spectrum may represent a serious loss (ROSENBLUTH [1960]).

The important types of radiation losses from free particles are due to

bremsstrahlung and cyclotron radiation. The former arise from the acceleration by Coulomb collisions, and the latter are due to the Larmor motion in a magnetic field. In addition, the excitation radiation and bremsstrahlung even from very small fractions of impurities in the plasma give rise to losses which far exceed those from hydrogen and its isotopes (see e.g. POST [1960]).

The spectra of radiation from particles bound in the atoms constitute a suitable diagnostic tool in astrophysics as well as in plasma experiments. Bremsstrahlung and cyclotron radiation from free particles can also be used to explore the properties of a plasma and the magnetic field in which it is immersed.

2. The Field of a Moving Charge

The wave equations (2.17) and (2.18) of Chapter 2 lead to a solution in vacuo which represents the propagation of electromagnetic waves. As a further step it is natural to consider how such waves are generated, i.e. how they originate from the source which consists of moving charges. We start with the simpliest case of a charge in vacuo. In presence of dense matter the situation becomes much more complicated and we shall only touch this in a simple consideration of Čerenkov radiation in § 4.

2.1. THE POTENTIALS OF A MOVING CHARGE

The field of a charge distribution σ associated with the current density $\boldsymbol{j}$ is given by the retarded potentials (2.19) and (2.20) in the Lorentz gauge. Care is necessary when these formulae are used to deduce the field from a point charge. For instance, the integral of equation (2.20) cannot simply be put equal to the value of q/r at the retarded time $t^* = t - R^*/c$. This is so, because we have to insert different values of the retarded time for each source point ρ^*. One therefore has to transform expressions (2.19) and (2.20) into integrals over the charge elements of the distribution σ, before a transition to a point charge can be made.

For this purpose, suppose all charge elements of a certain distribution σ to be rigidly connected with each other and to move at the same velocity $\boldsymbol{w}(t^*)$ at a given retarded time t^*. Consider a spherical shell of surface area S, thickness $\mathrm{d}R^*$, and situated at a distance R^* from the field point P given in Figure 10.1. An element of this shell has the volume $\mathrm{d}V^* = \mathrm{d}S\,\mathrm{d}R^*$. The real contribution to the integral (2.20) is given by the charge which a spherical light wave meets when it contracts with velocity c and arrives at P at the time t. It will pass the outer surface of the shell at time $t^* = t - R^*/c$.

During the time $\mathrm{d}t = \mathrm{d}R^*/c$ which this light wave takes to pass the shell it would sweep over a charge $\sigma \mathrm{d}S\mathrm{d}R^* = \sigma \mathrm{d}V^*$, if there would be no simultaneous motion of the charges. Since there is such a motion, however, a certain amount of charge will stream in or out through the inner surface of the shell during the time $\mathrm{d}t$ and will change the contribution to the integrals (2.19) and (2.20). At the source point ρ^* indicated in Figure 10.1 this amount becomes per unit area $\sigma \boldsymbol{w} \cdot \hat{\boldsymbol{R}}^* \mathrm{d}t = \sigma \boldsymbol{w} \cdot \hat{\boldsymbol{R}}^* \mathrm{d}R^*/c$, with the direction of

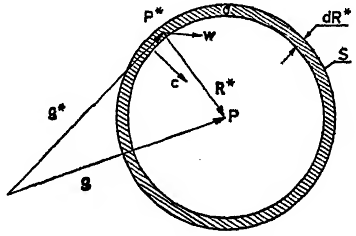


Fig. 10.1. A spherical light wave contracts with velocity c towards a field point P. It sweeps over a shell of thickness $\mathrm{d}R^*$ inside of which there is a charge distribution σ moving at velocity $\boldsymbol{w}$.

$\boldsymbol{R}^* = \rho - \rho^*$ along P*P. The charge element which is swept over by the light wave during time $\mathrm{d}t$, and which really contributes to the field at P, is therefore not $\sigma \mathrm{d}V^*$ but instead

$$\mathrm{d}Q = \sigma(1 - \boldsymbol{w} \cdot \hat{\boldsymbol{R}}^*/c)\mathrm{d}V^*. \tag{10.1}$$

The obtained expression (10.1) can now be inserted into equations (2.19) and (2.20) for a point charge q. We obtain

$$\boldsymbol{A}(\rho, t) = \frac{\mu_0}{4\pi}\left[\frac{q\boldsymbol{w}}{\rho - \boldsymbol{w} \cdot \rho/c}\right]_{t^*} \tag{10.2}$$

and

$$\phi(\rho, t) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\rho - \boldsymbol{w} \cdot \rho/c}\right]_{t^*}, \tag{10.3}$$

where the origin has been chosen at the position of the particle ($\rho^* = 0$, $\boldsymbol{R}^* = \rho$) and the values of the right hand members should be evaluated at the retarded time t^*. The potentials (10.2) and (10.3) were first obtained by LIENARD [1898] and WIECHERT [1900].

2.2. FIELD STRENGTHS OF A MOVING POINT CHARGE

From equations (2.8) and (2.10) the magnetic and electric fields can now be deduced by means of the expressions for the potentials $\boldsymbol{A}$ and ϕ. It then has to be observed that derivatives in equations (2.8) and (2.10) are taken with respect to time t and to the position $\boldsymbol{\rho}$ at the field point P, whereas expressions (10.2) and (10.3) are evaluated at the retarded time t^*. Thus, we have to relate the electromagnetic field at the field point P at time t to the position and motion of the charge at the source point P* at time t^*.

The motion of the particle at time t^* is given by $\rho(t^*)$ and $\boldsymbol{w}(t^*) = \partial\rho/\partial t^*$. The retarded time is defined by $\rho(t^*) = c(t - t^*)$ and accordingly

$$\frac{\partial\rho}{\partial t} = \frac{\partial\rho}{\partial t^*}\frac{\partial t^*}{\partial t} = -(\hat{\rho}\cdot\boldsymbol{w})\frac{\partial t^*}{\partial t} = c\left(1 - \frac{\partial t^*}{\partial t}\right) \tag{10.4}$$

or

$$\frac{\partial t^*}{\partial t} = \frac{1}{(1 - \boldsymbol{w}\cdot\rho/\rho c)}. \tag{10.5}$$

Since we have $\rho = c(t - t^*)$ the retarded time t^* also becomes a function of the coordinates of P. A deduction similar to that leading to equation (10.5) therefore yields

$$\nabla t^* = -\frac{\boldsymbol{\rho}}{c(1 - \boldsymbol{w}\cdot\rho/\rho c)}. \tag{10.6}$$

Insert now the potentials (10.2) and (10.3) into equations (2.8) and (2.10). After some deductions, where use is made of relations (10.4)–(10.6), the electric and magnetic fields at P become

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0 c^2}\frac{[\dot{\boldsymbol{w}} \times (\rho - \boldsymbol{w}\rho/c)] \times \rho + (1 - w^2/c^2)(\rho - \boldsymbol{w}\rho/c)}{(\rho - \boldsymbol{w}\cdot\rho/c)^3}, \tag{10.7}$$

$$\boldsymbol{B} = \rho \times \boldsymbol{E}/\rho c. \tag{10.8}$$

Here all quantities are understood to refer to the retarded time, ρ is the distance between the charge and the field point at this time and $\dot{\boldsymbol{w}} = \partial\boldsymbol{w}/\partial t^*$.

The electromagnetic field of equations (10.7) and (10.8) consists of one part which is proportional to the acceleration $\dot{\boldsymbol{w}}$ and decreases as $1/\rho$ at large distances, and one part which does not contain $\dot{\boldsymbol{w}}$ and which decreases as $1/\rho^2$ at large distances. The former is denoted as the *acceleration field* and is transverse in the sense that its components of $\boldsymbol{E}$ and $\boldsymbol{B}$ both are perpen-

dicular to $\boldsymbol{\rho}$. It predominates at great distances and it alone gives rise to radiation. The Poynting vector of this field decreases as $1/\rho^2$ at large distances and gives there a total energy flux which is independent of ρ, as expected. The latter part, which is denoted as the *velocity field*, represents a static contribution which reduces to the Coulomb field of a point charge when $\boldsymbol{w} = 0$. For $\boldsymbol{w} \neq 0$ it can be deduced from the Coulomb potential by means of a Lorentz transformation.

2.3. THE RADIATED ENERGY

Only the acceleration part of the electromagnetic field gives rise to radiation in vacuo. This is plausible since it can easily be shown that a particle cannot radiate if it moves at constant velocity in empty space. For the energy $\mathscr{E}$ of the particle we have, according to equations (9.28) and (9.29)

$$\mathscr{E}^2 = \frac{m^2c^4}{1 - w^2/c^2} = m^2c^4 + P^2c^2, \tag{10.9}$$

where $\boldsymbol{P}$ is the relativistic momentum. Differentiation of equation (10.9) leads to

$$\frac{\mathrm{d}P}{\mathrm{d}\mathscr{E}} = \frac{\mathscr{E}}{Pc^2} = \frac{[1 + (mc/P)^2]^{\frac{1}{2}}}{c}. \tag{10.10}$$

For a photon ($m = 0$) representing the electromagnetic field we would therefore have $\mathrm{d}P/\mathrm{d}\mathscr{E} = 1/c$ and for a particle of rest mass $m \neq 0$ we would instead obtain $\mathrm{d}P/\mathrm{d}\mathscr{E} > 1/c$. Thus, a freely moving particle cannot radiate since energy and momentum cannot be conserved simultaneously between the particle and the electromagnetic field alone. Consequently, external forces have to act on the particle to produce radiation. We shall give three examples of this in §§ 3, 4 and 5 of this Chapter.

We now calculate the instantaneous amount of energy radiated by the particle per unit of the time t^*. Suppose that the charge radiates during a time interval $\mathrm{d}t^*$. The time difference $\mathrm{d}t$ between the first and last signal observed at the field point is then $(1 - \boldsymbol{w} \cdot \boldsymbol{\rho}/\rho c)\mathrm{d}t^*$ according to equation (10.5). Due to Poynting's theorem the energy flow per unit of the time t and per unit area is $\boldsymbol{E} \times \boldsymbol{B}/\mu_0$. Consequently, the energy radiated by the charge per unit of the time t^* and per unit solid angle Y becomes

$$\frac{\mathrm{d}(\Delta P_{\mathrm{r}})}{\mathrm{d}Y} = \rho^2(1 - \boldsymbol{\rho} \cdot \boldsymbol{w}/\rho c)\boldsymbol{E} \times \boldsymbol{B}/\mu_0. \tag{10.11}$$

The modulus of the term $-\boldsymbol{\rho}\cdot\boldsymbol{w}/\rho c$ is less than unity and the latter oscillates sinusoidally when the particle gyrates around the external magnetic field. Insertion of expressions (10.7) and (10.8) for the acceleration field into (10.11) yields (LINHART [1960])

$$\frac{\mathrm{d}(\Delta P_r)}{\mathrm{d}Y} = \frac{q^2}{16\pi^2\varepsilon_0 c^3}\left[\frac{\dot{\boldsymbol{w}}^2}{(1-\boldsymbol{\rho}\cdot\boldsymbol{w}/\rho c)^3} + 2\frac{(\hat{\rho}\cdot\dot{\boldsymbol{w}})(\dot{\boldsymbol{w}}\cdot\boldsymbol{w}/c)}{(1-\boldsymbol{\rho}\cdot\boldsymbol{w}/\rho c)^4} - \frac{(1-w^2/c^2)(\hat{\rho}\cdot\dot{\boldsymbol{w}})^2}{(1-\boldsymbol{\rho}\cdot\boldsymbol{w}/\rho c)^5}\right]. \tag{10.12}$$

Integration of this expression with respect to the solid angle and over a large sphere results in the power

$$\Delta P_r = \frac{q^2}{6\pi\varepsilon_0 c^3}\left(\dot{w}_{(\parallel)}^2\gamma^6 + \dot{w}_{(\perp)}^2\gamma^4\right), \tag{10.13}$$

where $\dot{w}_{(\parallel)}$ and $\dot{w}_{(\perp)}$ are accelerations parallel with and perpendicular to the velocity $\boldsymbol{w}$. If we transform this expression to the rest system of the particle by the aid of equation (9.25) the result becomes

$$\Delta P_r = \frac{q^2}{6\pi\varepsilon_0 c^3}(\dot{\boldsymbol{w}}')^2. \tag{10.14}$$

We observe that the radiated power of (10.13) and (10.14) is proportional to the square of the electric charge. If the motions of a large number of particles is correlated in some way, the emitted radiation may therefore increase to very high intensities.

3. Bremsstrahlung

One of the accelerating forces which produce radiation arises from the Coulomb scattering in a dilute plasma. Consider single scattering events between two charged particles and neglect relativistic and quantum-mechanical effects. This requires the characteristic distance $|w^2/\dot{w}|$ for changes of the particle velocity to be much larger than the de Broglie wave length, h/mw, of the scattered particle, i.e.

$$|w^3/\dot{w}| \gg h/m. \tag{10.15}$$

For Coulomb scattering of an electron by an ion of charge Ze the acceleration is

$$\dot{w} \approx \frac{Ze^2}{4\pi\varepsilon_0 m_e \rho^2} \tag{10.16}$$

when the ion is assumed to be situated at the origin and its recoil is neglected. Further assume $w^2/c^2 \ll 1$. Since the Coulomb field is a central force field

the vector product between ρ and equation (2.36) can immediately be integrated and the result is that $\rho \times \mathrm{d}\rho/\mathrm{d}t$ becomes constant. This is the

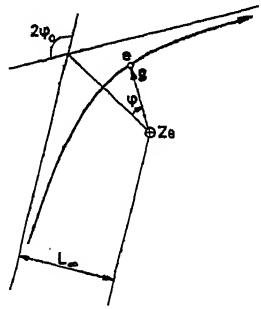


Fig. 10.2. Electron scattered by the electrostatic force of an ion.

familiar law of constant areas which, with the notation of Figure 10.2, becomes

$$\rho^2 \mathrm{d}\varphi/\mathrm{d}t = L_\infty w_\infty . \tag{10.17}$$

Here L_∞ denotes the impact parameter and w_∞ is the velocity of the electron at infinity.

Combination of equations (10.14), (10.16) and (10.17) yields the energy radiated during a collision,

$$\Delta\mathscr{E}_\mathrm{r} = \frac{Z^2 e^6}{96\pi^3 \varepsilon_0^3 m_\mathrm{e}^2 c^3 L_\infty w_\infty} \int_{-(\pi-\varphi_0)}^{(\pi-\varphi_0)} \frac{\mathrm{d}\varphi}{\rho^2} . \tag{10.18}$$

The total radiation emitted per unit volume is obtained from an integration for all electrons over all possible impact parameters L_∞ and angles φ_0. The largest contribution to this integral comes from the smallest values of L_∞ and φ_0. The smallest possible value of L_∞ is $h/2\pi m_\mathrm{e} w_\infty$ (cf. HEITLER [1954]). With n_i ions and n_e electrons per unit volume the power radiated per unit volume then becomes (cf. LINHART [1960] and ROSE and CLARK [1961])

$$P_\mathrm{rb} = \frac{\pi Z^2 e^6}{144 \varepsilon_0^3 m_\mathrm{e}^2 c^3 h} n_\mathrm{i} n_\mathrm{e} w_\infty . \tag{10.19}$$

In case of thermal equilibrium where an electron temperature T_e can be defined, equation (10.19) reduces to

$$P_\mathrm{rb} = 2.6 \times 10^{-40} n_\mathrm{i} n_\mathrm{e} Z^2 \sqrt{T_\mathrm{e}} \text{ watts/m}^3 . \tag{10.20}$$

This agrees quite well with a rigorous quantum mechanical deduction from which the numerical factor of equation (10.20) should be changed to 1.7×10^{-40} (HEITLER [1954], ROSE and CLARK [1961]).

Our present results indicate that the bremsstrahlung radiation increases rapidly with the density of particles and with the charge number of ions, but is a slow function of temperature. Due to its larger mass an ion experiences much smaller accelerations by Coulomb scattering than electrons of the same temperature. Therefore, the contribution from the power radiated by ions is negligible compared to that given by equation (10.20), at least in a plasma which is not too far from thermal equilibrium.

4. Čerenkov Radiation

In a dense medium the simultaneous interactions between many particles and between particles and radiation have to be taken into account. The situation then becomes more involved than that studied in the preceding paragraphs. Thus, multiple encounters and interactions at large distances will have a deciding influence on the radiation process.

A dense medium will not only be affected by the vacuum field from its moving charges. The medium itself will set up a secondary field represented by an equivalent dielectric constant which differs from that of vacuum. As a result the propagation velocity of light in the medium becomes smaller than c. An interesting consequence of this is that a high energy particle may move through the medium at velocities exceeding the local speed of light. This produces radiation in a form analogous to the propagation of a shock wave through a medium at supersonic speeds. During the motion momentum is steadily being absorbed by matter, in accordance with the conclusions of § 2.3. The effect was first observed by ČERENKOV [1937].

Special applications of Čerenkov radiation to problems in plasma physics have been considered by LINHART [1955, 1960] and by KIHARA *et al.* [1961].

5. Cyclotron Radiation

Radiation from acceleration of charges cannot only be produced by particle interactions but also by external fields. An example of this is cyclotron radiation which originates from the Larmor motion in a magnetic field.

The frequency spectrum of this radiation was studied at an early stage by SCHOTT [1912] and further contributions to the analysis were due to TZU [1948]. Applications to cosmical problems were suggested by POMERANCHUK

[1940] for the radiation by electrons in the earth's magnetic field. ALFVÉN and HERLOFSON [1950] proposed that radio star emission might be due to cosmic ray electrons in the trapping field of a star. Further astrophysical applications of cyclotron radiation were suggested by SHKLOVSKII [1953, 1958, 1960] and others for the emission from the Crab nebula. Since 1958 a great number of authors have connected the emission from the planet Jupiter with cyclotron emission. Their investigations are summarized in a recent review by CHANG [1962].

In thermonuclear research cyclotron radiation constitutes one of the most important energy losses. A detailed analysis of the problem was first performed by TRUBNIKOV [1958], TRUBNIKOV and KUDRYAVTSEV [1958] and TRUBNIKOV and BAZHANOVA [1958]. Further research is still going on up to this date.

5.1. THE POWER RADIATED FROM A TRANSPARENT PLASMA

A particle which gyrates in a magnetic field is accelerated in a direction perpendicular to its velocity $\boldsymbol{w}$. The acceleration $\dot{w}_{(\|)}$ defined in § 2.3 then vanishes. Insert the angles θ and φ defined in Figure 10.3 into equation

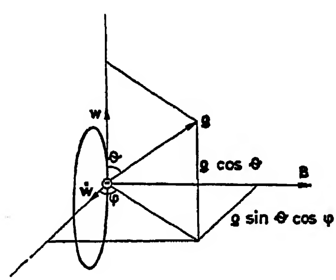


Fig. 10.3. Geometrical relations between velocity, acceleration and position vector of a field point for charge gyrating in a plane perpendicular to an external magnetic field $\boldsymbol{B}$.

(10.12) for the power radiated per unit solid angle. In the special case where the particle has no velocity component along the magnetic field follows that

$$\frac{\mathrm{d}(\Delta P_{\mathrm{rc}})}{\mathrm{d}Y} = \frac{q^2\dot{w}_{(\perp)}^2}{16\pi^2\varepsilon_0 c^3}\left[1 - \frac{(1-w^2/c^2)\cos^2\theta\sin^2\varphi}{[1-(w/c)\cos\theta]^2}\right]\Big/(1-(w/c)\cos\theta)^3. \tag{10.21}$$

Accordingly the radiation is anisotropic and has a maximum in the direction

along the velocity $\boldsymbol{w}$ of gyration, where $\theta = 0$ and $\varphi = 0$. In a highly relativistic case where w/c approaches unity the radiation is emitted in a very narrow cone around the direction of $\boldsymbol{w}$. As a result of this a nearly relativistic particle will emit pulses which are peaked about the time when the particle is approaching the observer. This leads to the emission of higher harmonics, as will also become clear from § 5.2.

The total emitted power measured in the laboratory system for a particle which spirals in the magnetic field is given by equation (10.13) with $\dot{w}_{(\parallel)} = 0$. When $E = 0$ the solution of (9.28) for the velocity of gyration yields $\dot{w}^2_{(\perp)} = \omega_g{}^2 w_\perp^2/\gamma^2$, where $\omega_g = qB/m$ is the non-relativistic value of the gyration frequency and $\boldsymbol{w}_\perp$ is the perpendicular component of the velocity. Consequently the total power emitted by a particle becomes

$$\Delta P_{rc} = \frac{q^2}{6\,\pi\varepsilon_0 c^3}\,\omega_g^2\gamma^2 w_\perp^2 = \frac{q^2}{6\,\pi\varepsilon_0 c^3}\,\omega_g^2 (P_\perp/m)^2\,, \qquad (10.22)$$

where $P_\perp$ is the perpendicular component of the relativistic momentum vector.

In an ionized gas in thermal equilibrium we can define a temperature T of the particles which have a Maxwellian velocity distribution. In the relativistic case the latter is given by the distribution function (CHAPMAN and COWLING [1952])

$$f(\boldsymbol{P}) = [4\pi m^2 ckTK_2(mc^2/kT)]^{-1} \exp\left[-(mc^2/kT)(1 - w^2/c^2)^{-\frac{1}{2}}\right] \quad (10.23)$$

with the modified Bessel function $K_2(mc^2/kT)$ of second order. The total power radiated per unit volume by particles of density n is obtained from an integration of (10.22) over the velocity distribution. According to Trubnikov and Kudryavtsev

$$P_{rc} = \frac{q^2\omega_g^2}{3\pi\varepsilon_0 c}\left(\frac{kT}{mc^2}\right) n \left(1 + \frac{5}{2}\,\frac{kT}{mc^2} + \ldots\right). \qquad (10.24)$$

Due to their smaller mass electrons will radiate much more than ions at a given temperature. The power radiated by electrons is

$$\mathrm{P}_{rc} = 5.3 \times 10^{-24} n_e B^2 T_e (1 + 4.2 \times 10^{-10} T_e + \ldots)\ \text{watts/m}^3. \qquad (10.25)$$

In a quasi-neutral deuterium plasma where ions and electrons have nearly the same temperatures and where the magnetic energy density $B^2/2\mu_0$ is of the same order of magnitude as the total pressure $2nkT$, the radiation power P_{rc} can be written as

$$P_{rc} \approx 3.7 \times 10^{-52} n^2 T^2 \ \text{watts/m}^3, \qquad (10.26)$$

as far as orders of magnitude are concerned. The ratio between this power and that arising from bremsstrahlung is according to equation (10.20)

$$P_{rc}/P_{rb} \approx 1.4 \times 10^{-12} T^{3/2}, \quad 4\mu_0 nkT/B^2 \approx 1, \quad Z = 1 \quad \cdot \qquad (10.27)$$

This ratio increases rapidly with temperature and passes the value unity at about 7×10^7 °K.

The present results are valid for a transparent plasma where the radiation emitted from every electron escapes out of its volume without interacting with the plasma. At thermonuclear temperatures in the range above 10^8 °K the cyclotron radiation losses would then become enormous. There would at least not exist any possibility of balancing them by the energy created in a power generator which uses deuterium as fuel. The losses given by equation (10.25) would even be of importance to the energy balance in a mixture of tritium and deuterium, where the reaction rate at these temperatures is more than ten times larger than in a pure deuterium plasma.

Fortunately, only a fraction of the radiated power predicted by equation (10.24) escapes from a plasma under thermonuclear conditions. The reason for this is that the radiation emitted from a single electron will interact with the surrounding electrons of the plasma. As a result, part of the radiation will be reabsorbed before it finds its way out of the plasma.

In a purely non-relativistic case a theoretical treatment would become comparatively easy. At the densities prevailing in a thermonuclear device the plasma frequency $\omega_p = (e^2 n/\varepsilon_0 m_e)^{\frac{1}{2}}$ is usually greater than the gyro frequency ω_e of electrons. For non-relativistic electrons the cyclotron radiation would then not be able to escape freely out of the plasma, and it would cause little worry as regards the energy balance. However, at thermonuclear temperatures the difficulty in the problem lies in the fact that even slightly relativistic particles, in the temperature range around 10^8 °K, will behave in a more complicated way as far as cyclotron radiation is concerned. Thus, the relativistic motion will produce higher harmonics and the spectral distribution and the absorption of the radiation become rather involved. We shall treat these questions in the coming sections and shall also discuss the possibilities by which the reabsorption of the radiation can be increased.

There still exists a difficult question in this connexion which we shall not be able to clarify here but the importance of which makes a short comment necessary. It concerns the non-thermal radiation which arises on account of instabilities, wave phenomena and other co-operative effects. By such phenomena the motions of the plasma particles become correlated. Since

ΔP_r is proportional to q^2 as shown by (10.13), this increases the emitted radiation to intensities far above those of a thermal plasma where the motions of individual particles are entirely uncorrelated. For further discussions on non-thermal radiation the reader is referred to a review by BEKEFI and BROWN [1961].

5.2. SPECTRAL DISTRIBUTION EMITTED FROM A SINGLE PARTICLE

We now study the spectrum of radiation emitted from a relativistic particle which spirals in a homogeneous magnetic field as shown in Figure 10.4. The

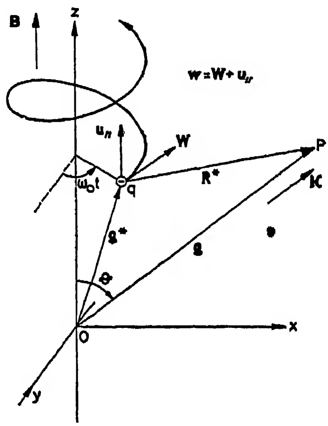


Fig. 10.4. Charged particle spiralling in a homogeneous magnetic field $\boldsymbol{B}$. The coordinate system is orientated in such a way that the radiation received at the field point P is given by the wave number κ in the xz plane. κ forms the angle θ with the z axis.

solution of the equation of motion (9.28) for the particle is for $\boldsymbol{E} = 0$

$$\rho^* = ((W/\omega_0) \sin \omega_0 t, \quad (W/\omega_0) \cos \omega_0 t, \quad u_{\|} t), \tag{10.28}$$

where $\omega_0 = qB/m\gamma(w)$, $w^2 = W^2 + u_{\|}^2$ and W and $u_{\|}$ are constant velocities.

Following TRUBNIKOV [1958] we start with the expression (10.2) for the vector potential generated by the particle:

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\left[\frac{q\boldsymbol{w}}{R^* - \boldsymbol{w}\cdot\boldsymbol{R}^*/c}\right]_{t^*}. \tag{10.29}$$

Evaluation should take place at the retarded time $t^* = t - R^*/c$. A Fourier expansion of this potential is defined by

$$\boldsymbol{A} = \int_{-\infty}^{+\infty} \boldsymbol{A}_\omega e^{i\omega t} d\omega, \quad \boldsymbol{A}_\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \boldsymbol{A}\, e^{-i\omega t} dt. \tag{10.30}$$

Now orientate the coordinate system in such a way that we can introduce the wave number

$$\boldsymbol{\kappa} = \kappa\,(\sin\theta, 0, \cos\theta) = \omega\hat{\boldsymbol{\kappa}}/c \tag{10.31}$$

in the wave zone, at large distances ρ from the origin O of Figure 10.4. We can then write

$$R^* \approx \hat{\boldsymbol{\kappa}}\cdot\boldsymbol{R}^* = \hat{\boldsymbol{\kappa}}\cdot(\boldsymbol{\rho} - \boldsymbol{\rho}^*) \tag{10.32}$$

in the wave zone for a field point P, where $\rho \gg \rho^*$. In this zone we ignore the small fluctuation introduced by $\boldsymbol{w}\cdot\boldsymbol{R}^*/c$ in the denominator of equation (10.29) and set $R^* = \rho$, but we preserve the fluctuation in $\boldsymbol{w}$, because its effect appears in lowest order (ROSE and CLARK [1961]). The components $\boldsymbol{A}_\omega$ are obtained from the Fourier transform and become

$$\boldsymbol{A}_\omega \approx \frac{\mu_0 q \exp(-\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{\rho})}{8\pi^2\rho}\int_{-\infty}^{+\infty}\boldsymbol{w}(t)\exp[\mathrm{i}(\boldsymbol{\kappa}\cdot\boldsymbol{\rho}^* - \omega t)]\mathrm{d}t. \tag{10.33}$$

From equations (10.28), (10.31) and from $\boldsymbol{w} = \mathrm{d}\boldsymbol{\rho}^*/\mathrm{d}t$ follows that

$$\boldsymbol{A}_\omega = \frac{\mu_0 q \exp(-\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{\rho})}{8\pi^2\rho}\int_{-\infty}^{+\infty}(W\cos\omega_0 t,\ -W\sin\omega_0 t,\ u_{||})$$

$$\times\exp\left[\mathrm{i}\,\frac{\omega}{c}\left(\frac{W}{\omega_0}\sin\theta\sin\omega_0 t + u_{||}t\cos\theta - ct\right)\right]\mathrm{d}t. \tag{10.34}$$

The result can be rewritten by means of the expansion

$$\mathrm{e}^{\mathrm{i}\chi\sin\varphi} = \sum_{n=-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}n\varphi}J_n(\chi) \tag{10.35}$$

for the Bessel functions J_n. Further use the relations

$$J_{n-1}(\chi) + J_{n+1}(\chi) = 2(n/\chi)J_n(\chi), \tag{10.36}$$

and

$$2J_n' \equiv 2\mathrm{d}J_n(\chi)/\mathrm{d}\chi = J_{n-1}(\chi) - J_{n+1}(\chi). \tag{10.37}$$

With the expansion

$$\delta(t - t_1) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp[\mathrm{i}\omega(t - t_1)]\mathrm{d}\omega \tag{10.38}$$

of the delta function we arrive after some deductions at the expressions

$$A_{\omega x} = \frac{\mu_0 c q \exp(-\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{\rho})}{4\pi\rho}\sum_{n=-\infty}^{+\infty}\frac{1 - (u_{||}/c)\cos\theta}{\sin\theta}J_n\left(\frac{\omega W}{\omega_0 c}\sin\theta\right)\delta_n, \tag{10.39}$$

$$A_{\omega y} = \frac{\mu_0 c q \exp(-\mathrm{i}\kappa\cdot\rho)}{4\pi\rho}\sum_{n=-\infty}^{+\infty}(-\mathrm{i})\frac{W}{c}J_n'\left(\frac{\omega W}{\omega_0 c}\sin\theta\right)\delta_n, \tag{10.40}$$

$$A_{\omega z} = \frac{\mu_0 c q \exp(-\mathrm{i}\kappa\cdot\rho)}{4\pi\rho}\sum_{n=-\infty}^{+\infty}(u_{||}/c)J_n\left(\frac{\omega W}{\omega_0 c}\sin\theta\right)\delta_n, \tag{10.41}$$

where J_n and J_n' are functions of the quantity within the last bracket of equations (10.39)—(10.41) and

$$\delta_n = \delta\left(\omega\frac{u_{||}}{c}\cos\theta - \omega + n\omega_0\right). \tag{10.42}$$

Observe that the delta function only gives contributions for a set of frequencies ω_n determined by equation (10.42):

$$\omega_n = \frac{n\omega_0}{1-(u_{||}/c)\cos\theta}. \tag{10.43}$$

The radiation field therefore consists of a spectrum of discrete frequencies given by this equation.

According to Poynting's theorem the total power radiated per unit area in the direction of κ is

$$|\boldsymbol{E}\times \operatorname{curl}\boldsymbol{A}/\mu_0| = (\omega/\mu_0)|\boldsymbol{A}\times(\kappa\times\boldsymbol{A})| = c|\kappa\times\boldsymbol{A}|^2/\mu_0, \tag{10.44}$$

since $\kappa\cdot\boldsymbol{A}=0$ and $\omega=\kappa c$. Analogously, we now define an energy of the n-th harmonic which is radiated per unit solid angle Y and per unit of ω during the whole course of events from $t=-\infty$ to $t=+\infty$:

$$\left(\frac{\mathrm{d}^2\mathscr{E}_{rc}}{\mathrm{d}Y\mathrm{d}\omega}\right)_n = 4\pi\rho^2 c\,|\,\kappa\times\boldsymbol{A}_{\omega n}|^2/\mu_0. \tag{10.45}$$

Here $\boldsymbol{A}_{\omega n}$ represents the n-th harmonic in equations (10.39)–(10.42).

Consider an oscillation of frequency ω_n during the time interval t_0 defined by

$$f(t) = \cos\omega_n t \text{ for } |t| < \tfrac{1}{2}t_0,\quad f(t)=0 \text{ for } |t| > \tfrac{1}{2}t_0. \tag{10.46}$$

We then have

$$f(t) = \int_{-\infty}^{+\infty} g(\omega)\exp(\mathrm{i}\omega t)\mathrm{d}\omega \tag{10.47}$$

and

$$g(\omega) = (1/2\pi)\int_{-\frac{1}{2}t_0}^{\frac{1}{2}t_0}\exp[\mathrm{i}(\omega_n-\omega)t]\mathrm{d}t = \frac{\sin[\frac{1}{2}(\omega_n-\omega)t_0]}{\pi(\omega_n-\omega)}. \tag{10.48}$$

When ω tends to ω_n the right hand member of (10.48) approaches $t_0/2\pi$. Suppose that, at the same time, the duration t_0 of the signal tends to infinity. It then follows from (10.38) and from the second member of (10.48) that the duration of the radiation period can be expressed as $2\pi\delta_n$ for the n-th harmonic. Dividing (10.45) by this quantity and substituting expressions (10.39) –(10.42) into it, we finally obtain

$$I_\omega^{(n)}(\theta) = \frac{q^2\omega^2}{8\pi^2\varepsilon_0 c}\left[\left(\frac{\cos\theta - u_\parallel/c}{\sin\theta}\right)^2 J_n^2\left(\frac{\omega W}{\omega_0 c}\sin\theta\right) + \right.$$
$$\left. + \left(\frac{W}{c}\right)^2 J_n'^2\left(\frac{\omega W}{\omega_0 c}\sin\theta\right)\right]\cdot\delta_n, \quad (n = 1, 2, \ldots) \tag{10.49}$$

for the radiation power emitted by the n-th harmonic per unit solid angle and per unit of the frequency ω. The term for $n = 0$ vanishes and terms with $n < 0$ can be included in the expressions (10.49) from symmetry reasons. The total power of radiation is

$$\sum_{n=1}^{\infty} I_n = \sum_{n=1}^{\infty}\int_0^{\pi}\int_{-\infty}^{+\infty} I_\omega^{(n)}(\theta) 2\pi\sin\theta\, d\theta\, d\omega \tag{10.50}$$

which can be shown to agree with the earlier result (10.22) (cf. TRUBNIKOV [1958]).

A detailed examination of equation (10.49) shows that for $w/c \ll 1$ the intensity of the harmonics declines rapidly as a function of the harmonic number n. For w/c approaching unity, on the other hand, the spectrum extends up to a considerable number of harmonics.

5.3. THE EMISSION COEFFICIENT

It is not obvious that the generated power density of cyclotron radiation in a plasma can be computed merely from a superposition of the contributions from all electrons, as obtained for an individual charge radiating in vacuo. These contributions may namely interfere with each other in a dense plasma. However, a closer examination of the problem by DRUMMOND and ROSENBLUTH [1960] (see also ROSE and CLARK [1961]) shows that it is in fact allowable to treat the motions of the electrons at thermonuclear conditions as if they were uncorrelated. This is possible when the frequencies of the harmonics of the radiated spectrum are definitely above the plasma frequency $\omega_p = (e^2 n/\varepsilon_0 m_e)^{\frac{1}{2}}$ (see also BEARD [1959]).

There are two physical arguments for the motions to become uncorrelated.

Firstly, electrons are within each other's Debye spheres of radii $(kT\varepsilon_0/ne^2)^{\frac{1}{2}}$ during times of order $1/\omega_p$. Their motions will then be uncorrelated for longer times when an average is taken over many cycles of gyration. Secondly, the kinetic energy of the particles is large compared to that of the electromagnetic radiation field. Therefore the kinetic energy will randomize the particle motion quite quickly and cooperative motions will be suppressed.

We now study the emission in a direction perpendicular to the magnetic field and put $\theta = \frac{1}{2}\pi$. Then, equations (10.39)–(10.42) reduce to

$$A_{\omega x} = \frac{\mu_0 cq \exp(-\mathrm{i}\kappa \cdot \rho)}{4\pi\rho} \sum_{n=-\infty}^{+\infty} J_n(\omega W/\omega_0 c)\delta_n(n\omega_0 - \omega) = A_{\omega z}c/u_{||}, \tag{10.51}$$

$$A_{\omega y} = \frac{\mu_0 cq \exp(-\mathrm{i}\kappa \cdot \rho)}{4\pi\rho} \sum_{n=-\infty}^{+\infty} (-\mathrm{i})\frac{W}{c} J_n'(\omega W/\omega_0 c)\delta_n(n\omega_0 - \omega). \tag{10.52}$$

From equation (10.49) we obtain a radiation power

$$I_\omega(\tfrac{1}{2}\pi) \equiv I_\omega^{(||)}(\tfrac{1}{2}\pi) + I_\omega^{(\perp)}(\tfrac{1}{2}\pi)$$
$$= \frac{q^2\omega^2}{8\pi^2\varepsilon_0 c}\sum_{n=1}^{\infty} [(u_{||}/c)^2 J_n^2(\omega W/\omega_0 c) + (W/c)^2 J_n'^2(\omega W/\omega_0 c)]\delta_n(n\omega_0 - \omega). \tag{10.53}$$

Here notice that terms with J_n correspond to $A_{\omega z}$. They represent ordinary waves which have an electric field component $\boldsymbol{E} = -\partial \boldsymbol{A}/\partial t$ that is parallel with the external magnetic field. Terms with J_n' correspond to $A_{\omega y}$ and represent extraordinary waves where $\boldsymbol{E}$ is perpendicular to the same field. These two types give rise to the powers $I_\omega^{(||)}(\frac{1}{2}\pi)$ and $I_\omega^{(\perp)}(\frac{1}{2}\pi)$, as defined by equation (10.53).

The emission coefficients of the plasma for these two wave types are now deduced from an integration of the radiated power of equation (10.53) over the relativistic distribution (10.23) of momenta. They are

$$\eta_\omega^{(||,\perp)}(\tfrac{1}{2}\pi) = \iiint I_\omega^{(||,\perp)}(\tfrac{1}{2}\pi) f(\boldsymbol{P})\mathrm{d}P_x\,\mathrm{d}P_y\,\mathrm{d}P_z \tag{10.54}$$

watts per unit solid angle, per unit volume and per unit of ω in the transverse direction. Alternative values in (10.54) are separated by a comma.

Especially for a weakly relativistic case, which is of interest in research on thermonuclear fusion, combination of equations (10.54), (10.53) and (10.23) yields after some deductions (cf. TRUBNIKOV [1958]):

$$\eta_\omega^{(\|,\perp)}(\tfrac{1}{2}\pi) \approx \frac{e^2 n_e \mu_r^{3/2} \omega_e (2\omega_e/\omega)^{\frac{1}{2}}}{32\pi^3 \varepsilon_0 c} \sum_{n>\omega/\omega_e}^{\infty} \left\{ \left(\frac{n^2\omega_e^2}{\omega^2} - 1 \right)^{\frac{1}{2}} \left(\frac{n - \omega/\omega_e}{n + \omega/\omega_e} \right)^n \right.$$

$$\left. \times \left[\frac{n^2 - \omega^2/\omega_e^2}{2\omega^3/\omega_e^3}, 1 \right] \exp\left[-\mu_r \left(\frac{n\omega_e}{\omega} - 1 \right) + \frac{2\omega}{\omega_e} \right] \right\} \qquad (10.55)$$

for electrons of density n_e and with $\omega_e = eB/m_e$ and $\mu_r = mc^2/kT_e$. In the first square bracket the two alternative values refer to the ordinary and extra-ordinary waves, respectively. Numerical calculations of the spectrum (10.55) have been made by TRUBNIKOV [1958] and by HIRSCHFIELD *et al.* [1961]. The result is a series of harmonics which are broadened by a Doppler effect due to the relativistic change of the momentum of the particles. At higher frequencies the harmonics overlap even so strongly that the total emission forms a monotonically decreasing function of frequency.

5.4. THE ABSORPTION COEFFICIENT

We make the assumption of a local thermodynamic equilibrium. In other words, we assume that the emission and absorption of radiation in every small volume of the plasma are the same as if the whole system were in thermodynamic equilibrium. The emission and absorption coefficients will then be related by Kirchhoff's law. Thus, the absorption coefficients $\alpha_\omega^{(\|,\perp)}(\theta)$ are given by

$$I_0 \equiv \eta_\omega^{(\|,\perp)}(\theta)/\alpha_\omega^{(\|,\perp)}(\theta) = h\omega^3/8\pi^3c^2[\exp(h\omega/kT_e) - 1]. \qquad (10.56)$$

Here we shall pay special attention to the direction $\theta = \frac{1}{2}\pi$ normal to the magnetic field.

In cases of practical interest $h\omega/kT_e \ll 1$ for cyclotron radiation. As an example, a field strength B of 5 V · sec/m² and a temperature T_e corresponding to 5×10^4 eV yields $h\omega/kT_e \approx 10^{-8}$. Further, it can be shown that the width of the spectral lines will exceed the distance between the lines when $\omega/\omega_e \gg (\frac{1}{2}\mu_r)^{3/2}$ and the spectrum then becomes nearly continuous. In cases of practical interest this also happens for all harmonics except for the first few ones.

Since the higher harmonics are responsible for the main radiation loss, it is sufficient to consider the continuous part of the spectrum. By using (10.56) and after approximating the series of (10.55) by an integral TRUBNI-

KOV [1958] obtains with the method of steepest descent:

$$\alpha_\omega^{(\|,\perp)}(\tfrac{1}{2}\pi) = 3\omega_p^2(\pi\mu_r)^{\frac{1}{2}}(2c\omega_e\varkappa_r)^{-1}\exp\left[-\mu_r(\varkappa_r^{\frac{1}{3}} - 1 + 9/20\varkappa_r^{\frac{1}{3}})\right]$$
$$\times\,[\mu_r^{-1}\varkappa_r^{-\frac{1}{3}}, 1],\ \varkappa_r = 9\omega kT_e/2\omega_e mc^2 \gg 1 \tag{10.57}$$

and

$$\alpha_\omega^{(\|,\perp)}(\tfrac{1}{2}\pi) = \omega_p^2\mu_r(\pi\omega_e/2\omega)^{\frac{1}{2}}(c\omega_e)^{-1}(e\omega/2\mu_r\omega_e)^{\omega/\omega_e}[\mu_r^{-1}, 1],\ \varkappa_r \ll 1. \tag{10.58}$$

Here ω_p is the plasma frequency. In both cases of equations (10.57) and (10.58) and for weakly relativistic electrons the ordinary wave $(_\|)$ will be attenuated much less than the extra-ordinary wave $(_\perp)$.

The emission coefficient has so far been calculated from the energy radiated by a distribution of particles. An equivalent method for its determination has been developed by DRUMMOND and ROSENBLUTH [1960] who start from the collisionless Boltzmann equation. They deduce the absorption coefficient for transverse electromagnetic waves directly from this equation and determine the emission coefficient from Kirchhoff's law.

The angular dependence of the absorption coefficient has been considered by DRUMMOND and ROSENBLUTH [1960] and by BEARD and BAKER [1961].

5.5. RADIATION FROM A PLANE SLAB

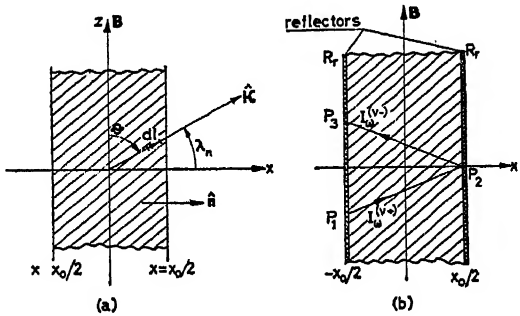


Fig. 10.5. Radiation from a plasma slab of thickness x_0 immersed in a magnetic field $\boldsymbol{B}$, parallel with the surfaces of the slab. (a) Definition of the angles θ and λ_n. (b) Slab bounded by reflectors with reflection coefficient R_r.

We now consider the radiation from the slab of Figure 10.5a of thickness x_0 which has its surfaces parallel with a homogeneous magnetic field $\boldsymbol{B}$. The latter is generated by external sources. A plane electromagnetic wave emitted in the direction θ will then have its wave normal $\hat{\kappa}$ inclined at an

angle λ_n to the normal $\hat{n}$ of the slab surface. Introduce the coordinate x along $\hat{n}$ as shown in Figure 10.5.

When the plasma is in radiative equilibrium the equation of energy transfer becomes

$$\frac{\partial}{\partial l} I_\omega^{(\nu)}(\theta) = \eta_\omega^{(\nu)}(\theta) - \alpha_\omega^{(\nu)}(\theta) I_\omega^{(\nu)}(\theta), \ (\nu = {}_\parallel, {}_\perp), \tag{10.59}$$

where the coordinate l is in the direction of propagation and superscript (ν) stands for the two directions $({}_\parallel, {}_\perp)$ of polarization. The emission and absorption coefficients are related by equation (10.56), where we still observe that $h\omega/kT_e \ll 1$ in cases of practical interest. Introduce this relation into equation (10.59) and integrate with respect to l for given directions θ and λ_n. The result is

$$I_\omega^{(\nu+)}(\theta, x) = I_0 + c_+ \exp\left[-\alpha_\omega^{(\nu)}(\theta)x/\cos\lambda_n\right] \tag{10.60}$$

for radiation in the positive x direction and

$$I_\omega^{(\nu-)}(\theta, x) = I_0 + c_- \exp\left[\alpha_\omega^{(\nu)}(\theta)x/\cos\lambda_n\right] \tag{10.61}$$

for radiation in the negative x direction. In equations (10.60) and (10.61) c_+ and c_- are constants of integration.

Now consider the situation of Figure 10.5b where the slab is bounded by reflectors at the surfaces $x = \pm \frac{1}{2}x_0$. Radiation which starts from the surface at P_1 will traverse the plasma and will be damped. It then reaches P_2 where the fraction R_r is reflected, traverses the plasma again in the negative x direction, and reaches P_3. From reasons of symmetry we can then impose two boundary conditions on the solutions (10.60) and (10.61), namely

$$I_\omega^{(\nu+)}(\theta, -\tfrac{1}{2}x_0) = R_r I_\omega^{(\nu-)}(\theta, -\tfrac{1}{2}x_0), \tag{10.62}$$

and

$$I_\omega^{(\nu-)}(\theta, \tfrac{1}{2}x_0) = R_r I_\omega^{(\nu+)}(\theta, \tfrac{1}{2}x_0). \tag{10.63}$$

This determines the constants c_+ and c_-. The intensity I_{eff} which actually escapes from the plasma is given by that transmitted through the reflector at P_2, i.e.,

$$\begin{aligned} I_{\text{eff}}^{(\nu)}(\omega, \theta, \lambda_n, x_0, R_r) &= (1 - R_r) I_\omega^{(\nu+)}(\theta, \tfrac{1}{2}x_0) \\ &= I_0(1 - R_r)\frac{1 - \exp\left[-\alpha_\omega^{(\nu)}x_0/\cos\lambda_n\right]}{1 - R_r\exp\left[-\alpha_\omega^{(\nu)}x_0/\cos\lambda_n\right]}. \end{aligned} \tag{10.64}$$

Equation (10.64) has been deduced by TRUBNIKOV [1958] and by DRUMMOND and ROSENBLUTH [1960]. These authors also suggested the use of reflectors to cut down the losses of cyclotron radiation from the plasma. For frequencies ω where the reflection coefficient R_r is very close to unity we see that the effective radiation intensity is cut down far below the level I_0 which would be reached for black-body radiation without reflectors.

In a weakly relativistic gas we have earlier found that the higher harmonics contain a considerable amount of energy. For these harmonics the absorption coefficient unfortunately turns out to be quite small and $\alpha_\omega^{(\nu)} x_0 \ll 1$. It then follows that equation (10.64) reduces to

$$I_{\mathrm{eff}} \approx I_0(\alpha_\omega^{(\nu)} x_0/\cos \lambda_n) \cdot \left[1 + \frac{R_r(\alpha_\omega^{(\nu)} x_0/\cos \lambda_n)}{1 - R_r}\right]^{-1}. \tag{10.65}$$

There exist two possibilities:

(i) If the absorption is very small and the reflectors are not extremely good the value of $R_r(\alpha_\omega^{(\nu)} x_0/\cos \lambda_n)/(1 - R_r)$ will become much smaller than unity and I_{eff} approaches the value which is obtained from equation (10.64) with $R_r = 0$, i.e. for a plasma without reflectors. This situation has been investigated by HIRSCHFIELD *et al.* [1961] who have presented values of the absorption coefficient within a large range of the spectrum.

(ii) When the reflectors are extremely good and R_r is so close to unity that even $R_r(\alpha_\omega^{(\nu)} x_0/\cos \lambda_n)/(1 - R_r) \gg 1$ for the higher frequencies of the spectrum, the effective intensity approaches $I_0(1 - R_r)$ and there is a strong reduction in the radiation losses.

5.6. BALANCE BETWEEN RADIATION LOSSES AND POWER PRODUCTION

We finally investigate the total radiation power Π_{rc} emitted per unit area of a plasma slab. It is then necessary to integrate the intensities (10.64) for both directions of polarization. This should be done at the surface of the plasma slab, over a half sphere and over the whole range of frequencies:

$$\Pi_{rc} = \int_0^\infty \int_{\lambda_n=0}^{\frac{1}{2}\pi} (I_{\mathrm{eff}}^{(\parallel)} + I_{\mathrm{eff}}^{(\perp)}) \cos \lambda_n \, \mathrm{d}Y \, \mathrm{d}\omega. \tag{10.66}$$

If the plasma were completely transparent it would radiate a power P_{rc} per unit volume as given by equation (10.24). For a volume element of the slab of length x_0 and area $\mathrm{d}S$ the radiated power would then become $P_{rc} x_0 \mathrm{d}S$ in absence of absorption and of reflectors. In reality the effective radiation

loss from the same element is instead $2\Pi_{rc}\mathrm{d}S$. We can therefore define an effective transparency coefficient

$$K_{rc} = 2\Pi_{rc}/P_{rc}x_0. \tag{10.67}$$

When $\alpha_\omega^{(v)}x_0$ approaches zero and $(1 - R_r)$ differs from zero it follows from equations (10.65), (10.66) and (10.56) that

$$2\Pi_{rc}(\alpha_\omega^{(v)}x_0 = 0) = 2\int_0^\infty\int_{\lambda_n=0}^{\frac{1}{2}\pi} x_0[\eta_\omega^{(\|)}(\theta) + \eta_\omega^{(\perp)}(\theta)]\mathrm{d}Y\,\mathrm{d}\omega. \tag{10.68}$$

This expression is equal to the total power $P_{rc}x_0$ radiated inside the slab per unit area $\mathrm{d}S$. Consequently, the transparency coefficient K_{rc} then approaches unity, as it should. Thus, we write

$$\begin{aligned}
K_{rc}\int_0^\infty\int_{\lambda_n=0}^{\frac{1}{2}\pi} x_0[\eta_\omega^{(\|)}(\theta) + \eta_\omega^{(\perp)}(\theta)]\mathrm{d}Y\mathrm{d}\omega \\
= \int_0^\infty\int_{\lambda_n=0}^{\frac{1}{2}\pi} I_0(1-R_r)\cos\lambda_n \left\{\frac{1-\exp[-\alpha_\omega^{(\|)}x_0/\cos\lambda_n]}{1-R_r\exp[-\alpha_\omega^{(\|)}x_0/\cos\lambda_n]}\right. \\
\left. + \frac{1-\exp[-\alpha_\omega^{(\perp)}x_0/\cos\lambda_n]}{1-R_r\exp[-\alpha_\omega^{(\perp)}x_0/\cos\lambda_n]}\right\}\mathrm{d}Y\mathrm{d}\omega.
\end{aligned} \tag{10.69}$$

In particular, suppose that the radiation losses are exactly balanced by a power production P_s per unit volume. The latter may be created by sources inside the plasma, such as thermonuclear reactions. The powers $P_sx_0\mathrm{d}S$ and $2\Pi_{rc}\mathrm{d}S$ should then become equal, and this leads to a critical transparency coefficient

$$K_{rc}{}^* = P_s/P_{rc}. \tag{10.70}$$

The results of this paragraph can be used to calculate the critical thickness $x_0{}^*$ of the slab when the power production just balances the cyclotron radiation losses. TRUBNIKOV [1958] has done this under the assumption that all the radiation leaks out of the system for frequencies ω above that corresponding to $\alpha_\omega^{(v)}x_0 = 1$. Further, from an estimation of the angular distribution with respect to θ Trubnikov concludes that an approximate expression can be used for K_{rc}. The latter is equal to half the value which is obtained when the transverse coefficients $\alpha_\omega^{(v)}(\frac{1}{2}\pi)$ of equations (10.57) and (10.58) are substituted into equation (10.69) instead of $\alpha_\omega^{(v)}(\theta)$. Equations (10.70), (10.24), (10.69), (10.57), (10.58) and (10.56) then form a system

from which x_0^* can be calculated. Numerical values have been given by TRUBNIKOV and KUDRYAVTSEV [1958] which suggest that the critical dimension of the plasma slab becomes quite large under thermonuclear conditions. An example of these calculations can be mentioned where $B = 1\ \mathrm{V \cdot sec/m^2}$, $T_i = T_e$, the plasma pressure is equal to the magnetic energy density $B^2/2\mu_0$, and where the particle density n is of the order of $10^{19}\ \mathrm{m^{-3}}$. Then, the corresponding dimension x_0^* turns out to be somewhat more than 20 m in the case of a deuterium plasma.

The extremely large size of fusion reactors which this result predicts is discouraging and has induced a number of investigators such as BEARD [1959], DRUMMOND and ROSENBLUTH [1960, 1961], and HIRSCHFIELD *et al.* [1961] to make further studies of the problem and of possible methods by which the radiation losses can be reduced. The situation has not yet been settled but according to recent papers by DRUMMOND and ROSENBLUTH [1961] and DRUMMOND [1961] the critical size can possibly be brought down to less than a meter by means of reflectors. Whether the conditions can be improved by using superconducting materials in order to increase the reflection coefficient cannot be decided here, but may be worth further investigations.

It should finally be pointed out that the reaction rate in a mixture of tritium and deuterium is more than ten times larger than that in a pure deuterium plasma. With such a mixture a balance of the radiation losses should be possible, also for a moderate size x_0^* of the plasma.

LIST OF SYMBOLS

(MKSA-units are used. Reference is made to equations or chapters where symbol first appears)

$\boldsymbol{A}$	Magnetic vector potential (2.8)
$\boldsymbol{A}^*$	Modified vector potential in rotating system (7.28)
(A_ν)	Electromagnetic four-potential (9.17)
$\boldsymbol{A}_\omega$	Fourier components of vector potential (10.30)
$\boldsymbol{a}$	Vector connecting the centre of gyration with the position of a charged particle; a is the radius of gyration or Larmor radius (2.80; 3.6); or the 'relativistic' radius of gyration (9.34)
$\boldsymbol{a}_{\mathrm{CM}}$	Centre of mass of guiding centra (7.43)
$a_{1\mathrm{f}}$, a_{10}	Coefficients in eq. (4.63)
$a_{\mu\nu}$	Coefficients of orthogonal transformation (9.20)
$\boldsymbol{B}$	Magnetic field strength (2.1)
$\boldsymbol{B}^*$	Modified magnetic field in rotating system (7.31)
B_{m}	Modulus of field strength at magnetic mirror (6.18)
b	Coefficient in eq. (9.34)
$\boldsymbol{C}$	Position vector of centre of gyration (3.4)
C, C', C''	Notations for different frames of reference (Ch. 9)
C_{b}	Function defined in eq. (8.17)
$\boldsymbol{C}_\nu$	Coefficients describing position of particle in Ch. 3, § 1 ($\nu = 1, 2, \ldots$)
$C_{\|\|}$	Invariant of longitudinal compression, associated with J (5.61)
$C_{1\mathrm{f}}$, C_{10}	Coefficients in eq. (4.63)
c	Velocity of light (2.9979×10^8 m/s)
c_ν	Constants used in particular problems; any subscript ($_\nu$)
c_+, c_-	Constants of integration (10.60; 10.61)
D, D'	Operators defined in eqs. (5.41) and (5.42)
D_w	Operator defined in eq. (5.9)

d_c	Mean distance between particles during collisions (Ch. 7, § 4.2)
d_1	Half the distance between centra of two current leads (Ch. 7, § 2.1 (ii))
$\boldsymbol{E}$	Electric field strength (2.1)
$\boldsymbol{E}^*$	Modified electric field in rotating system (7.30)
e	Charge of proton (1.602×10^{-19} As)
$\mathscr{E}$	Energy (9.29)
$\mathscr{E}_{rc}$	Energy of n-th harmonic due to cyclotron radiation (10.45)
$\Delta\mathscr{E}_r$	Energy radiated in a collision (10.18)
$\boldsymbol{F}$	External force field (2.36)
F	Density of phase points in $6N$-dimensional phase space (5.1)
$F_{\mu\nu}$	Electromagnetic field tensor (9.18)
$F_{\mu\nu}^{(0)}$, $F_{\mu\nu}^{(1)}$	Zero and first order parts of $F_{\mu\nu}$ (Ch. 9, § 4.4)
f	Distribution function in six-dimensional phase space (5.2)
$f(t)$	Function defined in eq. (10.47)
f_ν	Distribution function in νth order approximation (Ch. 5, § 1.4)
$f^{(\nu)}$	Contribution to f in νth order (5.44)
$\boldsymbol{f}_{\mu\nu}$	Force due to collisions (7.48)
G	Generating function defined in Ch. 2, § 3.1
G_{ie}	Parameter defined in connexion with eqs (8.33) and (8.50)
G_1 G_2 G_3 G_4	Generating functions defined in Ch. 2, § 3.1
$G_1(t_1)$	Function defined in eq. (4.77)
$\boldsymbol{g}$	Acceleration due to gravity (8.3)
$g(\omega)$	Function defined in eq. (10.48)
g_B	Function defined in eq. (6.31)
g_ν	Functions defined in connexion with eqs. (5.46)
H	Hamiltonian (2.52)
H'	Transformed Hamiltonian in guiding centre approach (3.61)
$H_\parallel$	Hamiltonian corresponding to longitudinal motion (4.7)
h	Planck's constant (6.625×10^{-34} kg m^2/s)

h_k	Scale factor associated with generalized coordinate q_k (7.1)
I	Total electric current (2.113)
I_n	Intensity of n-th harmonic (10.50)
I_0	Radiation intensity (10.56)
$I_\omega^{(n)}$	Spectral intensity of radiation (10.49)
$I_{\mathrm{eff}}^{(\nu)}$	Effective radiation intensity escaping from plasma (10.64)
$I_\omega^{(\parallel,\perp)}(\frac{1}{2}\pi)$	Radiation power of ordinary and extra-ordinary waves (10.53)
$\mathscr{I}$	Value of integral (4.94)
i	Imaginary unit, $(-1)^{\frac{1}{2}}$
J	Longitudinal (or second adiabatic) invariant (4.8)
J_k	Action angle variable associated with k-direction (2.63)
J^*, $J_k{}^*$	Action integrals of nearly periodic system (2.71)
$J_\parallel{}^*$	Longitudinal action integral (4.8)
$J_\perp{}^*$	Transverse action integral (4.12)
$\boldsymbol{j}$	Electric current density (2.2)
K_{rc}	Transparency coefficient (10.67)
$K_{\mathrm{rc}}{}^*$	Critical transparency coefficient (10.70)
K_1, K_2, K_3, K_4, K_5	Characteristic parameters of drift motion (3.30)
$K_\parallel$, $K_\perp$	Longitudinal and transverse mean energies per particle (3.45)
k	Boltzmann's constant (1.380×10^{-23} kg m^2/s^2 degree)
k_x, k_y, k_z	Wave numbers in rectangular coordinates (Ch. 8, § 2)
k_1, k_2, k_3	Characteristic parameters of particle orbit (2.43)
L	Lagrangian (2.47)
L_c	Characteristic length $\lvert \chi \rvert / \lvert \nabla\chi \rvert$ of a certain quantity χ
$\tilde{L}_{\mathrm{cmin}}$	Length of steepest variation of perturbation (Ch. 8, § 2.4)
L_x, L_y, L_z	Wave lengths in rectangular coordinate system (Ch. 8, § 2)
L_ν	Notation for magnetic field lines; $\nu = 0, 1, 2, \ldots$ (Ch. 2, § 3)
L_∞	Impact parameter (10.17)
$\boldsymbol{l}$	Length coordinate along a certain specified direction; d$\boldsymbol{l}$ line element (2.25)
l_0	Characteristic length in eq. (2.95)

l'	Dimensionless arc length (2.109)
$\boldsymbol{M}$	Equivalent magnetic moment or first adiabatic invariant (2.83; 3.16; 9.62; 9.77)
$\boldsymbol{M}_p$	Moment of magnetic dipole (2.107)
m	Mass of particle; m_i and m_e are ion and electron masses (2.36)
m_φ	Wave number in φ direction (8.56)
N	Number of particles (Ch. 5, § 1.1); unperturbed particle density (Ch. 8, § 2)
N'	Density gradient (positive or negative) in Ch. 8, § 2
n	Degrees of freedom (Ch. 2, § 3.2), particle density (3.40), exponential factor (8.56) or harmonic number (10.43)
$\hat{\boldsymbol{n}}$	Unit vector along normal direction (2.27)
n_1	Quantity specified in eq. (4.54)
$O(\chi)$	Order of magnitude of the quantity χ
$\boldsymbol{P}$	Relativistic momentum (9.28)
P	Unperturbed scalar pressure (6.51)
$P_\parallel$, $P_\perp$	Unperturbed longitudinal and transverse pressures (6.54)
$P_\perp^*$	Perpendicular relativistic momentum as defined in connexion with eq. (9.77)
$P(t_1)$	Function defined by eq. (4.72)
P_{cl}	Probability of particle loss by close encounters (7.56)
P_D	Probability of particle loss by distant encounters (7.57)
P_e, P_i	Unperturbed pressures of electrons and ions (Ch. 8, § 2.2)
P_{rb}	Power radiated as bremsstrahlung per unit volume (10.19)
P_{rc}	Power radiated as cyclotron radiation per unit volume (10.24)
P_s	Power generated in plasma per unit volume (10.70)
ΔP_r	Power radiated by single particle (10.13)
ΔP_{rc}	Cyclotron radiation power of single particle (10.22)
p	Scalar pressure (5.25)
p_j, p_k	Generalized momenta of particle (2.51)
p_1, p_2, p_3	Generalized momenta in guiding centre approach (3.56)
$p_\parallel$, $p_\perp$	Longitudinal and transverse pressures (5.21)

$\boldsymbol{Q}$	Heat flux vector (5.26); $\boldsymbol{Q}_{(k)}$ is the part associated with the velocity component w_k
Q	Electric charge (10.1)
Q_{p}	Particle density in (α, β, J, M) space (4.51)
q	Charge of particle, positive or negative (2.36)
q_j, q_k; q_m, q_n	Generalized coordinates of particle (2.49); Ch. 7, § 2.1
q_1, q_2, q_3	Generalized coordinates in guiding centre approach (3.57)
$\boldsymbol{R}$	Vector indicating radius of curvature (3.20)
$\boldsymbol{R}^*$	Difference between position vector $\boldsymbol{\rho}$ of field point and position vector $\boldsymbol{\rho}^*$ of source point (2.19)
R	Radius of curvature positive or negative (8.2); mirror ratio (6.22)
$R(t)$	Mirror ratio (6.20)
R_{eff}	Effective mirror ratio (7.61)
R_{m}	Maximum mirror ratio (6.20)
R_{r}	Reflection coefficient (10.62)
r	Radial distance from axis in cylindrical coordinate system (Ch. 2, § 4.2)
S	Area; dS element of area (2.27)
S_ν	Coefficients describing position of particle in Ch. 3, § 1 ($\nu = 1, 2 \ldots$)
S_{b}	Function defined in eq. (8.17)
s	Coordinate along magnetic field line; ds is line element (Ch. 2, § 1.2)
s_B	Function defined in eq. (6.29)
s_{m}	Half the distance along field line between two magnetic mirrors (6.24)
T	Temperature (10.20)
t	Time (2.1)
t_{c}	Characteristic time $\mid \chi \mid / \mid \mathrm{d}\chi/\mathrm{d}t \mid$ of a certain quantity χ (2.40)
t_{cf}	Characteristic time scale of electromagnetic field as seen from a coordinate system following particle orbit (Table 4.1)
t_{cl}	Time for close encounters (7.50)
t_{D}	Deflection time for distant encounters (7.51)
t_{g}	Gyro or Larmor period, $2\pi/\omega_g$ (4.4)

t_ω	Characteristic time of frequency variation (4.67)
t_1	Time variable defined in eq. (4.69)
$t_{\parallel}$	Period of longitudinal oscillations between two magnetic mirrors (4.9)
$t_{\perp}$	Period of transverse drift around a magnetic field configuration (4.13)
$\boldsymbol{U}$	Space part of four velocity (9.46)
$\boldsymbol{U}_g$	Group velocity (Ch. 6, § 3)
$\boldsymbol{U}_n$	Convection velocity of density distribution (5.67)
U_p	Phase velocity (6.52)
U_s	Speed of sound (6.53)
$U_{s\parallel}, U_{s\perp}$	Parallel and transverse speeds of sound (6.57)
U_t	Time part of four velocity (9.46)
U_{tr}	Thermal energy per unit mass (5.30)
$U_{(k)}$	Part of thermal energy per unit mass associated with velocity component w_k (5.26)
U_ν	Four-velocity components of guiding centre (9.44); ($\nu = 1, 2, 3, 4$)
$U_\nu^{(0)}, U_\nu^{(1)}$	Zero and first order parts of U_ν (9.66)
$\boldsymbol{u}$	Velocity of guiding centre (3.9; 3.16)
$\boldsymbol{u}'$	Velocity satisfying eq. (5.38)
$\boldsymbol{u}_B$	Magnetic gradient drift (3.24); mean magnetic gradient drift (8.46)
$\boldsymbol{u}_E$	Electric field drift (3.26)
$\boldsymbol{u}_F$	External force drift (3.23)
$\boldsymbol{u}_f, \boldsymbol{u}_f$	Velocities defined in connexion with eqs. (8.32) and (8.47)
$\boldsymbol{u}_g$	Gravitation drift (Ch. 8, § 2.3)
$\boldsymbol{u}_{gi}, \boldsymbol{u}_{ge}$	Gravitation drifts of ions and electrons (Ch. 8, § 2.1)
u_K	Velocity defined in connexion with eq. (8.34)
u_{Kz}	Velocity defined in connexion with eq. (8.52)
$\boldsymbol{u}_m$	Transverse inertia drift (3.25)
$\boldsymbol{u}_p$	Polarization drift (3.27)
u_1	Variable specified in eq. (4.93)
$\boldsymbol{u}_{\parallel}, \boldsymbol{u}_{\perp}$	Longitudinal and transverse drifts of guiding centre (3.17; 3.18; 3.19)
$\bar{\boldsymbol{u}}_{\parallel}$	Mean value of longitudinal drift for certain distribution of particles (3.45)

$\tilde{u}_{\|\|}$	"Thermal" part of longitudinal drift (3.45)
$u_{\|\|}$	The variable $\mathrm{d}s/\mathrm{d}t$, with sign included (Ch. 4, § 1.1)
V	Volume; $\mathrm{d}V$ element of volume (2.19)
V_A	Alfvén velocity (6.52)
$\boldsymbol{V}_F$, $\boldsymbol{V}_\mathrm{f}$	Velocities of moving contours (2.27; 2.35)
$V_{\alpha\beta}$	Modulus of mean velocity in $\alpha\beta$ plane (4.60)
$\boldsymbol{V}_\perp$	Velocity carrying a particle as specified in Fig. 4.5
$\boldsymbol{v}$	Macroscopic velocity or mass velocity (3.42)
$\boldsymbol{v}_\mathrm{g}$	Part of mass velocity due to the gyration of particles (5.54)
v_1	Variable specified in Ch. 4, § 2.3
$\boldsymbol{W}$	Velocity of gyration (Larmor motion), perpendicular to magnetic field in first order; $\boldsymbol{W} = \mathrm{d}\boldsymbol{a}/\mathrm{d}t$ (2.82; 3.9)
W	Hamilton's characteristic function (2.60)
W_3	First order approximation of velocity of gyration in eq. (3.64)
$\boldsymbol{w}$	Total velocity of particle; $\boldsymbol{w} = \mathrm{d}\boldsymbol{\rho}/\mathrm{d}t$ (2.36)
$\boldsymbol{w}^*$	Velocity of particle observed in rotating coordinate system (7.22)
$\boldsymbol{w}'$	Transformed velocity (5.37)
$\tilde{\boldsymbol{w}}$	Deviation of particle velocity from its mean value (5.19)
w_∞	Velocity at infinity (10.17)
X_ν	Coordinates of guiding centre in four-space; $\nu = 1, 2, 3, 4$ (9.51)
x, y, z or x_k	Rectangular coordinates; $k = x, y, z$ (2.40; 2.48)
x_0	Thickness of plasma slab (10.62)
x_0^*	Critical thickness (Ch. 10, § 5.6)
x_ν	Coordinates of particle in four-space; $\nu = 1, 2, 3, 4$ (9.7)
$\tilde{x}_\nu$	Small oscillating contribution to particle motion (9.51)
$\boldsymbol{Y}$	Velocity carrying particle as specified in Ch. 4, § 1.5
Y	Solid angle (10.11)
v_1	Variable defined in eq. (4.85)
Z	Charge number of ion (10.16)
α, β	Transverse coordinates specifying magnetic field line (2.21); definition modified in Ch. 3, § 3
$\alpha_1, \ldots, \alpha_n$	Canonical momenta in transformed representation (Ch. 2, § 3.2)

α_{ie}	Frequency defined in connexion with (8.20)
α_{ν}	Coefficient in eq. (9.34)
$\alpha_{\omega}^{(\|\|,\perp)}$	Absorption coefficients with respect to frequency for ordinary and extra-ordinary waves (10.56)
β_{ν}	Coefficient in eq. (9.34)
Γ	Stability parameter defined in eqs (8.26), (8.36) and (8.51)
Γ'	Parameter defined in eq. (8.27)
γ	Relativistic factor, $(1 - w^2/c^2)^{-\frac{1}{2}}$ (9.3)
γ_c	Collision parameter giving rate of change towards isotropy (6.44)
$2\gamma'$	Dimensionless angular momentum at infinity (2.110)
δ	Parameter defined in eq. (4.68), or any subscript (9.58)
$\delta(\chi)$	Delta function of χ (10.38)
$\delta_{\mu\nu}$	Kronecker's delta; $\delta_{\mu\nu} = 1$ for $\mu = \nu$ and $\delta_{\mu\nu} = 0$ for $\mu \neq \nu$ (5.21)
δ_{ν}	Components of vector given by eq. (9.53)
ε	Ratio between particle mass m and charge q. Represents also dimensionless "smallness parameter" expressing the ratios between the Larmor radius and the characteristic lengths, or between the gyro time and the characteristic time of the electromagnetic field (3.3; 9.59)
ε_{eq}	Equivalent dielectric constant (or inductive capacity) (3.51)
ε_0	Dielectric constant (or inductive capacity) in vacuo ($(1/36\pi) \times 10^{-9}$ As/Vm)
ζ	Variable defined by eqs. (2.139) and (4.62)
ζ_{α}	Variable given by eq. (9.52); ζ_{α}^* is the complex conjugate of ζ_{α}.
ζ_1	Variable given by $\zeta_1 = x + iy$ (2.138)
ζ_2	Variable introduced in eq. (4.76)
η	Angle of rotation in four-space (9.10)
η_{ν}	Coefficient in eq. (9.34)
$\eta_{\omega}^{(\|\|,\perp)}$	Emission coefficients with respect to frequency ω for ordinary and extra-ordinary waves (10.54)
Θ	Periodic part of variables describing motion of charged particle (2.64)
θ	Angle (3.20; 6.52) and azimuthal coordinate (10.21)

θ_B	Angle defined in Fig. 5.1.b
ϑ	Periodic part of variables associated with Larmor motion of a charged particle (3.6)
$\vartheta_{\|\|}$	Periodic part of variables associated with the longitudinal oscillations of a charged particle (4.8)
ι	Angle of rotational transform (Ch. 7, § 3.2)
$\boldsymbol{\kappa}$	Wave number (Ch. 6, § 3; 10.31)
$\boldsymbol{\varkappa}$	Function associated with magnetic compression and defined in eq. (8.45)
$\varkappa$	Total compression factor (6.28)
$\varkappa_r$	Parameter defined in connexion with eq. (10.57)
$\varkappa_{\|\|}, \varkappa_{\perp}$	Longitudinal and transverse compression factors (6.25; 6.27)
Λ	Ratio measuring deviations from adiabatic invariance (4.66)
Λ_D	Parameter defined in eq. (7.53)
Λ_e, Λ_i	Parameters defined by eq. (8.54)
λ	Parameter defined in eq. (9.35)
λ_c, λ_s	Amplitudes (8.27)
λ_e, λ_i	Spatial perturbations of ion and electron clouds (8.11)
λ_n	Angle defined in Fig. 10.5
μ	Any subscript (9.18)
μ_r	Parameter defined in connexion with eq. (10.55)
μ_0	Magnetic permeability in vacuo ($4\pi \times 10^7$ Vs/Am)
ν	Any subscript (9.18)
ξ	Displacement of fluid element (6.7)
ξ_v	Coefficient in eq. (9.34)
Π_{rc}	Power generated as cyclotron radiation per unit area of plasma slab (10.66)
π	Pressure tensor (5.19)
$\boldsymbol{\rho}$	Position vector of field point (2.19) and of particle (3.4)
$\boldsymbol{\rho}^*$	Position vector of source point (2.19)
$\boldsymbol{\rho}'$	Position vector in transformed frame (9.1)
σ	Electric charge density (2.3) or any subscript (9.57)
σ_s	Electric surface charge density (8.21)
σ_v	Components of vector given by (eq. (9.53)
τ	Proper time (9.30)
τ_f, τ_0	Proper times defined in Ch. 9, § 4. 3

τ_w	Decay time due to velocity diffusion (7.60)
τ_ω	Proper time defined in eq. (9.64)
τ_1	Variable introduced in eq. (4.79); $\tau_1 = \varepsilon t_1$
Φ	Magnetic flux (2.25); flux invariant or third adiabatic invariant (4.12)
ϕ	Electric potential (2.10)
ϕ_{eq}	Equivalent potential (7.7)
ϕ_g	Gravitation potential (2.37)
ϕ^*	Modified electric potential in rotating system (7.29)
φ	Meridional angle or angle in cylindrical coordinate system (5.39)
χ	Indicates any scalar quantity (2.13), or the modulus $\mid \chi \mid$ of any vector $\boldsymbol{\chi}$, unless something else is specified
Ψ	Total particle flux (3.41)
$\Psi(l_1 w)$	Function defined in eq. (7.54)
Ψ_u	Contribution to particle flux from guiding centre drift (3.41)
Ψ_W	Contribution to particle flux from gyration (3.41)
ψ_D	Function defined in eq. (7.55)
$\boldsymbol{\Omega}$	Angular velocity (7.20)
ω	Frequency of oscillation (4.62)
ω_e, ω_i	Frequency of gyration of electrons and ions (8.9)
ω_f, ω_0	Frequencies in final and initial states (4.63)
ω_g	Frequency of gyration (Larmor frequency) of any particle (2.72)
ω_n	Frequency of n-th harmonic (10.43)
ω_p	Plasma frequency (Ch. 10, § 5.3)
$\omega^{(0)}$	Zero order approximation of ω in eq. (9.67)
$\mathrm{d}\chi/\mathrm{d}t \equiv \dot{\chi}$	Total time derivative of χ in a coordinate system following the motion of a particle (2.36; 2.48)
$\mathrm{d}_\nu\chi/\mathrm{d}t$	Time derivatives $(\partial/\partial t + \boldsymbol{v}_\nu \cdot \nabla)\chi$ of χ in coordinate systems following the mass motions of ions or electrons; $\nu =$ i, e (Ch. 8, § 2.2)
∇	Nabla operator (2.10)
$\nabla_w \equiv \partial/\partial\boldsymbol{w}$	Operator in velocity space; $\nabla_w = (\partial/\partial w_x, \partial/\partial w_y, \partial/\partial w_z)$ in rectangular coordinates (5.9)
$\hat{\boldsymbol{\chi}}$	Unit vector $\boldsymbol{\chi}/\chi$ along $\boldsymbol{\chi}$ (2.27)
χ	Modulus $(\chi^2)^{\frac{1}{2}}$ of vector $\boldsymbol{\chi}$, unless someting else is spe-

	cified (2.81)
$\langle\chi\rangle$	Mean value (expectation value) of χ, except in velocity space (3.12)
$\bar{\chi}$	Mean value of χ in velocity space (5.12)
$\tilde{\chi}$	Perturbed part of χ (Ch. 8, § 2.2) or deviation of χ from an average (Ch. 5, § 1.3)
$\{\chi_1, \chi_2\}$	Poisson bracket of two quantities, χ_1 and χ_2 (2.59)

Superscripts and subscripts have the following meaning:

$(^*)$	Modified quantity or complex conjugate
$(')$	Transformed or dimensionless variable
$(_C)$	Value at centre of gyration
$(_c)$	Characteristic quantity
$(_e)$	Refers to electrons
$(_f)$	Final state
$(_i)$	Refers to ions
$(_j)$, $(_k)$	Any subscript
$(_{max})$, $(_{min})$	Maximum and minimum values
$(_0)$	Original state, initial value, or value in equatorial plane
$(_w)$	Value at bounding wall
$(_\delta)$, $(_\mu)$, $(_\nu)$, $(_\sigma)$	Any subscript
$(_\parallel)$, $(_\perp)$	Parallel with and perpendicular to a specified direction

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SUBJECT INDEX

COMMONLY USED EXPRESSIONS AND FIGURES

(These pages can be cut out of the book to be used as a separate pamphlet)

$$\operatorname{curl} \boldsymbol{E} = -\,\partial \boldsymbol{B}/\partial t \tag{2.1}$$

$$\operatorname{curl} \boldsymbol{B}/\mu_0 = \boldsymbol{j} + \varepsilon_0\, \partial \boldsymbol{E}/\partial t \tag{2.2}$$

$$\operatorname{div} \boldsymbol{E} = \sigma/\varepsilon_0 \tag{2.7}$$

$$\boldsymbol{B} = \operatorname{curl} \boldsymbol{A} \tag{2.8}$$

$$\boldsymbol{E} = -\nabla\phi - \partial \boldsymbol{A}/\partial t \tag{2.10}$$

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\,\mu_0 \boldsymbol{j} \tag{2.17}$$

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\,\sigma/\varepsilon_0 \tag{2.18}$$

$$\boldsymbol{A}(\rho, t) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{j}(\rho^*, t - R^*/c)}{R^*}\,\mathrm{d}V^* \tag{2.19}$$

$$\phi(\rho, t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(\rho^*, t - R^*/c)}{R^*}\,\mathrm{d}V^* \tag{2.20}$$

$$\boldsymbol{A} = \alpha\nabla\beta + \nabla\chi \tag{2.21}$$

$$\boldsymbol{B} = \nabla\alpha \times \nabla\beta \tag{2.22}$$

$$\boldsymbol{E} = -\nabla\left(\phi + \alpha\frac{\partial\beta}{\partial t}\right) + \frac{\partial\beta}{\partial t}\nabla\alpha - \frac{\partial\alpha}{\partial t}\nabla\beta \tag{2.23}$$

$$E_{\|} = -\frac{\partial}{\partial s}\left(\phi + \alpha\frac{\partial\beta}{\partial t}\right) \tag{2.24}$$

$$m\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t} = \boldsymbol{F} + q\boldsymbol{w} \times \boldsymbol{B} \tag{2.36}$$

$$\boldsymbol{F} = -\nabla(q\phi + m\phi_g) - q\frac{\partial \boldsymbol{A}}{\partial t} = q\boldsymbol{E} - m\nabla\phi_g \tag{2.37}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}mw^2) = \boldsymbol{F}\cdot\boldsymbol{w} \tag{2.38}$$

$$k_1\frac{\mathrm{d}^2\boldsymbol{\rho}'}{\mathrm{d}t'^2} = -k_2\nabla'\phi' - k_3\nabla'\phi_g' - \frac{\partial \boldsymbol{A}'}{\partial t'} + \frac{\mathrm{d}\boldsymbol{\rho}'}{\mathrm{d}t'}\times \mathrm{curl}'\,\boldsymbol{A}' \tag{2.42}$$

$$k_1 = mL_c/qA_ct_c;\quad k_2 = t_c\phi_c/A_cL_c;\quad k_3 = mt_c\phi_{gc}/qA_cL_c \tag{2.43}$$

$$L = \tfrac{1}{2}mw^2 - m\phi_g - q\phi + q\boldsymbol{w}\cdot\boldsymbol{A} \tag{2.47}$$

$$p_k = \frac{\partial}{\partial \dot{q}_k}L(q_j, \dot{q}_j, t) \tag{2.51}$$

$$H = \sum_k p_k\dot{q}_k - L(q_k, \dot{q}_k, t) \tag{2.52}$$

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \tag{2.56}$$

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} \tag{2.57}$$

$$p_k = \frac{\partial G_2}{\partial q_k};\ q_k' = \frac{\partial G_2}{\partial p_k'};\ H' = H + \frac{\partial G_2}{\partial t} \tag{2.58}$$

$$\frac{\mathrm{d}\chi}{\mathrm{d}t} = \frac{\partial\chi}{\partial t} + \sum_k\left(\frac{\partial\chi}{\partial q_k}\dot{q}_k + \frac{\partial\chi}{\partial p_k}\dot{p}_k\right) = \frac{\partial\chi}{\partial t} + \{\chi, H\} \tag{2.59}$$

$$p_k = mw_k + qA_k \tag{2.61}$$

$$H = \tfrac{1}{2}mw^2 + q\phi + m\phi_g \tag{2.62}$$

$$q_k = q_k[\Theta(t), t];\quad p_k = p_k[\Theta(t), t] \tag{2.64}$$

$$J^* = \sum_{k=1}^{n} J_k^* = \text{const.};\quad J_k^* = \oint\left[p_k\frac{\partial q_k}{\partial\Theta}\right]_t \mathrm{d}\Theta \tag{2.71}$$

$$\boldsymbol{a} = \frac{1}{\omega_g}\hat{\boldsymbol{B}}\times\boldsymbol{W}_0\cos\omega_g t + \frac{1}{\omega_g}\boldsymbol{W}_0\sin\omega_g t;\quad \boldsymbol{W}_0 = (w_{x0}, w_{y0}, 0) \tag{2.80}$$

$$\boldsymbol{W} = \mathrm{d}\boldsymbol{a}/\mathrm{d}t = \omega_g\boldsymbol{a}\times\hat{\boldsymbol{B}} \tag{2.82}$$

$$\boldsymbol{M} = -\tfrac{1}{2}mB(W_0/B)^2 \tag{2.83}$$

$$a \left| \frac{\partial B_j}{\partial x_k} \right| / |B_j| \ll 1 \tag{3.1}$$

$$\frac{1}{\omega_g} \left| \frac{dB_j}{dt} \right| / |B_j| \ll 1 \tag{3.2}$$

$$\varepsilon \frac{d\boldsymbol{w}}{dt} = \boldsymbol{F}/q + \boldsymbol{w} \times \boldsymbol{B}, \quad \varepsilon = m/q, \quad \boldsymbol{w} = d\boldsymbol{\rho}/dt \tag{3.3}$$

$$\begin{aligned} \boldsymbol{\rho}(t) &= \boldsymbol{C}(t) + \sum_{\nu=1}^{\infty} \varepsilon^{\nu}\{\boldsymbol{C}_{\nu}(t) \cos[\nu\vartheta(t)/\varepsilon] + \boldsymbol{S}_{\nu}(t) \sin[\nu\vartheta(t)/\varepsilon]\} \\ &\equiv \boldsymbol{C}(t) + \boldsymbol{a}(t) \end{aligned} \tag{3.6}$$

$$\boldsymbol{u} = d\boldsymbol{C}/dt; \quad \boldsymbol{W} = d\boldsymbol{a}/dt; \quad \boldsymbol{w} = \boldsymbol{u} + \boldsymbol{W} \tag{3.9}$$

$$m \frac{d\boldsymbol{u}}{dt} = \boldsymbol{F} + q\boldsymbol{u} \times \boldsymbol{B} - M\nabla B; \quad M = mW^2/2B \tag{3.16}$$

$$m\hat{\boldsymbol{B}} \cdot \frac{d\boldsymbol{u}}{dt} = \boldsymbol{F} \cdot \hat{\boldsymbol{B}} - M(\hat{\boldsymbol{B}} \cdot \nabla)B \tag{3.17}$$

$$\boldsymbol{u}_{\perp} = \left(\boldsymbol{F} - M\nabla B - m \frac{d\boldsymbol{u}}{dt}\right) \times \boldsymbol{B}/qB^2 \tag{3.18}$$

$$\frac{d\boldsymbol{u}}{dt} = u_{||} \frac{d\hat{\boldsymbol{u}}_{||}}{dt} + \hat{\boldsymbol{u}}_{||} \cdot \frac{du_{||}}{dt} + \frac{d\boldsymbol{u}_{\perp}}{dt} \tag{3.19}$$

$$\begin{aligned} d\hat{\boldsymbol{u}}_{||} &= \hat{\boldsymbol{u}}_{||} - \hat{\boldsymbol{u}}_{||0} = \hat{\boldsymbol{B}} - \hat{\boldsymbol{B}}_0 = (|\boldsymbol{R}|\, d\theta \hat{\boldsymbol{B}} \cdot \nabla)\hat{\boldsymbol{B}} \\ &= u_{||}(\hat{\boldsymbol{B}} \cdot \nabla)\hat{\boldsymbol{B}}\, dt \end{aligned} \tag{3.20}$$

$$(\hat{\boldsymbol{B}} \cdot \nabla)\hat{\boldsymbol{B}} = -\hat{\boldsymbol{B}} \times \operatorname{curl} \hat{\boldsymbol{B}} = (B\nabla_{\perp}B - \boldsymbol{B} \times \operatorname{curl} \boldsymbol{B})/B^2 \tag{3.21}$$

$$\begin{aligned} \boldsymbol{u}_{\perp} &= \left[\boldsymbol{F} - M(1 + 2u_{||}^2/W^2)\nabla B - m \frac{d\boldsymbol{u}_{\perp}}{dt}\right] \times \boldsymbol{B}/qB^2 \\ &\quad + (2Mu_{||}^2/W^2qB)(\operatorname{curl} \boldsymbol{B})_{\perp} \end{aligned} \tag{3.22}$$

$$\boldsymbol{u}_F = \boldsymbol{F} \times \boldsymbol{B}/qB^2 \quad \text{(external force drift)} \tag{3.23}$$

$$\boldsymbol{u}_B = [M(1 + 2u_{||}^2/W^2)/qB^2]\boldsymbol{B} \times \nabla B \quad \text{(magnetic gradient drift)} \tag{3.24}$$

$$\boldsymbol{u}_m = (m/qB^2)\boldsymbol{B} \times \frac{d\boldsymbol{u}_{\perp}}{dt} \quad \text{(transverse inertia drift)} \tag{3.25}$$

$$\boldsymbol{u}_E = \boldsymbol{E} \times \boldsymbol{B}/B^2 \quad \text{(electric drift)} \tag{3.26}$$

$$\boldsymbol{u}_\mathrm{p} = (m/qB^4)\boldsymbol{B} \times \left(\frac{\partial \boldsymbol{E}}{\partial t} \times \boldsymbol{B}\right) = (m/qB^2)\frac{\partial \boldsymbol{E}_\perp}{\partial t} \quad \text{(polarization drift)} \tag{3.27}$$

$$\dot{\vartheta}^2\boldsymbol{C}_1 + \dot{\vartheta}\boldsymbol{S}_1 \times \boldsymbol{B} = \varepsilon(2\dot{\vartheta}\dot{\boldsymbol{S}}_1 + \ddot{\vartheta}\boldsymbol{S}_1 - \boldsymbol{C}_1 \times \boldsymbol{B}) - \varepsilon\dot{\boldsymbol{C}} \times [(\boldsymbol{C}_1 \cdot \nabla)\boldsymbol{B}] - \varepsilon(\boldsymbol{C}_1 \cdot \nabla)\boldsymbol{F}/q + O(\varepsilon^2) \tag{3.32}$$

$$\dot{\vartheta}^2\boldsymbol{S}_1 - \dot{\vartheta}\boldsymbol{C}_1 \times \boldsymbol{B} = -\varepsilon(2\dot{\vartheta}\dot{\boldsymbol{C}}_1 + \ddot{\vartheta}\boldsymbol{C}_1 + \dot{\boldsymbol{S}}_1 \times \boldsymbol{B}) - \varepsilon\dot{\boldsymbol{C}} \times [(\boldsymbol{S}_1 \cdot \nabla)\boldsymbol{B}] - \varepsilon(\boldsymbol{S}_1 \cdot \nabla)\boldsymbol{F}/q + O(\varepsilon^2) \tag{3.33}$$

$$\boldsymbol{F} = -q\dot{\boldsymbol{C}} \times \boldsymbol{B} + O(\varepsilon); \quad \boldsymbol{F} \cdot \boldsymbol{B} = O(\varepsilon) \tag{3.34}$$

$$\boldsymbol{C}_1 \cdot \boldsymbol{B} = O(\varepsilon); \quad \boldsymbol{S}_1 \cdot \boldsymbol{B} = O(\varepsilon); \quad \boldsymbol{C}_1 \cdot \boldsymbol{S}_1 = O(\varepsilon) \tag{3.35}$$

$$C_1^2 = S_1^2 + O(\varepsilon) \tag{3.36}$$

$$\dot{\vartheta} = B + O(\varepsilon) \tag{3.37}$$

$$\boldsymbol{a} = \varepsilon\boldsymbol{C}_1 \cos\omega_\mathrm{g}t + \varepsilon\boldsymbol{S}_1 \sin\omega_\mathrm{g}t + O(\varepsilon) \tag{3.38}$$

$$\varepsilon\boldsymbol{C}_1 = \hat{\boldsymbol{B}} \times \boldsymbol{W}_0/\omega_\mathrm{g}; \quad \varepsilon\boldsymbol{S}_1 = \boldsymbol{W}_0/\omega_\mathrm{g} \tag{3.39}$$

$$m\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = (1 + \tfrac{1}{4}a^2\nabla_\perp^2)\boldsymbol{F} + q\boldsymbol{u} \times \boldsymbol{B} - M\nabla B + O(\varepsilon^2\boldsymbol{B}) + O(\varepsilon^3) \tag{3.40}$$

$$n\boldsymbol{v} = n\boldsymbol{u} + \operatorname{curl}(n\boldsymbol{M}/q) \tag{3.43}$$

$$K_\| = \tfrac{1}{2}m\overline{u_\|^2} = \tfrac{1}{2}m(\bar{u}_\|^2 + \overline{\tilde{u}_\|^2}); \quad K_\perp = \tfrac{1}{2}m\overline{W^2} \tag{3.45}$$

$$n\boldsymbol{v}_\| = n\bar{\boldsymbol{u}}_\| - nK_\perp(\operatorname{curl}\boldsymbol{B})_\|/qB^2 \tag{3.46}$$

$$n\boldsymbol{v}_\perp = n\boldsymbol{F} \times \boldsymbol{B}/qB^2 + n(K_\perp - 2K_\|)(\nabla B - \hat{\boldsymbol{B}} \times \operatorname{curl}\boldsymbol{B}) \times \boldsymbol{B}/qB^3 - \nabla(nK_\perp) \times \boldsymbol{B}/qB^2 - nm\frac{\overline{\mathrm{d}\boldsymbol{u}_\perp}}{\mathrm{d}t} \times \boldsymbol{B}/qB^2 \tag{3.47}$$

$$nm\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}\right]_\| = n\boldsymbol{F}_\| - (n/B)(K_\perp - 2K_\|)\nabla_\| B - \nabla_\|(2nK_\|) \tag{3.49}$$

$$\varepsilon_\mathrm{eq} = (n_\mathrm{i}m_\mathrm{i} + n_\mathrm{e}m_\mathrm{e})/B^2 \tag{3.51}$$

$$H' = \frac{1}{\varepsilon}H_{-1} + H_0 + \varepsilon H_1 + \ldots \tag{3.61}$$

$$H_{-1} = q\phi' + \frac{\alpha}{\varepsilon}\left(\frac{\partial \beta}{\partial t}\right) \tag{3.62}$$

$$J^* = \oint [m(\boldsymbol{W} + \boldsymbol{u}) + q\boldsymbol{A}]_t \frac{\partial}{\partial \vartheta}(\boldsymbol{C} + \boldsymbol{a})_t \, \mathrm{d}\vartheta = \text{const.} \tag{4.2}$$

$$m\left(\frac{\mathrm{d}\boldsymbol{u}_{||}}{\mathrm{d}t}\right) \cdot \hat{\boldsymbol{B}} \approx q\boldsymbol{E}_{||} - M\frac{\partial B}{\partial s}\hat{\boldsymbol{B}} \tag{4.6}$$

$$H_{||} = q\phi + q\alpha\frac{\partial \beta}{\partial t} + \tfrac{1}{2}mu_{||}^2 + MB \tag{4.7}$$

$$J_{||}{}^* \equiv J = \oint \left[p_{||}\frac{\partial s}{\partial \vartheta_{||}}\right]_t \mathrm{d}\vartheta_{||} = m\oint u_{||}\,\mathrm{d}s \approx \text{const.} \tag{4.8}$$

$$J = \oint p_{||}\,\mathrm{d}s = m\int_0^{t_{||}} u_{||}^2\,\mathrm{d}t = m\,\langle u_{||}^2\rangle\, t_{||} = \text{const.} \tag{4.9}$$

$$J = m\oint u_{||}(s')\mathrm{d}s'; \quad u_{||} = (2/m)^{\frac{1}{2}}\left(H_{||} - q\phi - q\alpha\frac{\partial \beta}{\partial t} - MB\right)^{\frac{1}{2}} \tag{4.24}$$

$$\langle\dot{\alpha}\rangle = -\frac{1}{q}\frac{\partial H_{||}}{\partial \beta} \tag{4.44}$$

$$\langle\dot{\beta}\rangle = \frac{1}{q}\frac{\partial H_{||}}{\partial \alpha} \tag{4.45}$$

$$\ddot{\zeta} + \omega^2(t)\zeta = 0 \tag{4.62}$$

$$M_0 = (q/2\omega_0)(\dot{x}^2 + \dot{y}^2)_0 = (q/2\omega_0)\langle|\dot{\zeta}_1|^2\rangle_0 \tag{4.65}$$

$$\frac{\partial n}{\partial t} + \operatorname{div}(n\boldsymbol{v}) = 0 \tag{5.17}$$

$$nm\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = n(\boldsymbol{F} + q\boldsymbol{v}\times\boldsymbol{B}) - \operatorname{div}\pi \tag{5.20}$$

$$\pi_{jk} = p_{||}\hat{B}_j\hat{B}_k + p_\perp(\delta_{jk} - \hat{B}_j\hat{B}_k) \tag{5.21}$$

$$\operatorname{div}\pi = \nabla_{||}p_{||} - \frac{p_{||} - p_\perp}{B}\nabla_{||}B + \nabla_\perp p_\perp + (p_{||} - p_\perp)(\hat{\boldsymbol{B}}\cdot\nabla)\hat{\boldsymbol{B}} \tag{5.24}$$

$$n\boldsymbol{v}_\perp = (n\boldsymbol{F} - \operatorname{div}\pi)\times\boldsymbol{B}/qB^2 - nm\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right]\times\boldsymbol{B}/qB^2 \tag{5.52}$$

$$\boldsymbol{v} \approx \boldsymbol{v}_{||} + \boldsymbol{F}\times\boldsymbol{B}/qB^2; \quad \boldsymbol{v}_{||} \approx \bar{\boldsymbol{u}}_{||}; \quad v_g \approx \mathrm{O}(\varepsilon) \tag{5.55}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla\right)(p/n^{5/3}) = 0 \tag{5.60}$$

$$\begin{aligned}
-\frac{\partial n}{\partial t} = &\left(\boldsymbol{F} \times \boldsymbol{B}/qB^2 - m\frac{\mathrm{d}_v \boldsymbol{v}}{\mathrm{d}t} \times \boldsymbol{B}/qB^2 + \boldsymbol{v}_{||}\right) \cdot \nabla n \\
&+ (2/qB^3)(\boldsymbol{B} \times \nabla B) \cdot \operatorname{div} \pi + n \operatorname{div}\left(\boldsymbol{F} \times \boldsymbol{B}/qB^2 - m\frac{\mathrm{d}_v \boldsymbol{v}}{\mathrm{d}t} \times \boldsymbol{B}/qB^2\right) \\
&+ (1/qB^2) \operatorname{div}(\boldsymbol{B} \times \operatorname{div} \pi) + nB(\hat{\boldsymbol{v}}_{||} \cdot \nabla)(v_{||}/B)
\end{aligned} \tag{5.62}$$

$$\begin{aligned}
\frac{\partial n}{\partial t} &+ \boldsymbol{U}_n \cdot \nabla n + n \operatorname{div}\left[\boldsymbol{u}_F + \frac{m}{qB^2}\boldsymbol{B} \times \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}_{||} + \boldsymbol{u}_F)\right] \\
&+ \frac{n}{qB^2}[2\hat{\boldsymbol{B}} \times \nabla B + (\operatorname{curl} \boldsymbol{B})_\perp] \cdot \nabla K_\perp \\
&+ \frac{n}{qB^2} \operatorname{div}\{(2K_{||} - K_\perp)[\hat{\boldsymbol{B}} \times \nabla B + (\operatorname{curl} \boldsymbol{B})_\perp]\} \\
&- \frac{2n}{qB^5}(2K_{||} - K_\perp)(\boldsymbol{B} \times \nabla B) \cdot (\boldsymbol{B} \times \operatorname{curl} \boldsymbol{B}) + nB(\hat{\boldsymbol{v}}_{||} \cdot \nabla)(v_{||}/B) = 0
\end{aligned} \tag{5.67}$$

$$\begin{aligned}
\frac{\partial n}{\partial t} - \frac{n}{B} \cdot \frac{\partial B}{\partial t} &- (1/q)\boldsymbol{F} \cdot [(n/B^2) \operatorname{curl} \boldsymbol{B} - \boldsymbol{B} \times \nabla(n/B^2)] \\
&+ \boldsymbol{v}_{||} \cdot \nabla n + nB(\hat{\boldsymbol{v}}_{||} \cdot \nabla)(v_{||}/B) = 0
\end{aligned} \tag{6.12}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}_F \cdot \nabla\right)(n/B^2) = 0; \quad \boldsymbol{v}_{||} = 0 \tag{6.13}$$

$$\frac{\mathrm{d}p_j}{\mathrm{d}t} = 0; \quad p_j = \text{const.} = p_{j0} = mh_{j0}^2 \dot{q}_{j0} + qh_{j0}A_{j0} \tag{7.6}$$

$$\begin{aligned}
\phi_{\mathrm{eq}} \equiv \tfrac{1}{2}m(w_m^2 + w_n^2) = H - H_0 + \tfrac{1}{2}mw_0^2 + m(\phi_{g0} - \phi_g) + q(\phi_0 - \phi) \\
- (1/2mh_j^2)[mh_{j0}w_{j0} + q(h_{j0}A_{j0} - h_jA_j]^2 \geqslant 0
\end{aligned} \tag{7.7}$$

$$\begin{aligned}
\phi_{\mathrm{eq}} = \tfrac{1}{2}mw_0^2 + m(\phi_{g0} - \phi_g) + q(\phi_0 - \phi) \\
- (1/2mr^2)[mr_0w_{\varphi 0} + q(r_0A_{\varphi 0} - rA_\varphi)]^2 \geqslant 0
\end{aligned} \tag{7.13}$$

$$\boldsymbol{w} = \boldsymbol{w}^* + \boldsymbol{\Omega} \times \boldsymbol{\rho}^* \tag{7.23}$$

$$m \frac{\mathrm{d}\boldsymbol{w}^*}{\mathrm{d}t} = q\boldsymbol{E}^* + q\boldsymbol{w}^* \times \boldsymbol{B}^* - m\nabla\phi_g \tag{7.32}$$

$$R_m = (\sin\theta)^{-2} + 2q(\phi_0 - \phi_m)/mW_0^2 \tag{7.40}$$

$$\boldsymbol{E}_0 \approx -(m_i/e)\boldsymbol{g} + (1/eN)(\operatorname{div}\pi_{i0})_\perp - \tfrac{1}{2}(m_i/e)\nabla(\boldsymbol{\Omega} \times \rho)^2 \tag{8.5}$$

$$\begin{aligned}-\frac{\partial n_\nu}{\partial t} = \operatorname{div}(n_\nu \boldsymbol{v}_\nu) = &-2\operatorname{div}\left[\frac{n_\nu}{\omega_\nu B}\boldsymbol{B} \times (\boldsymbol{v}_\nu \times \boldsymbol{\Omega})\right] \\ &-(\nabla\tilde{\phi} \times \boldsymbol{B})\cdot\nabla(n_\nu/B^2) + \frac{1}{B}\left(\frac{1}{\omega_\nu} - \frac{1}{\omega_i}\right)\{[\boldsymbol{g} + \tfrac{1}{2}\nabla(\boldsymbol{\Omega}\times\rho)^2]\times\boldsymbol{B}\}\cdot\nabla n_\nu \\ &+\frac{2}{q_\nu B^3}(\boldsymbol{B} \times \nabla B)\cdot\operatorname{div}\pi_\nu - \frac{1}{q_\nu B^2}\boldsymbol{B}\cdot\operatorname{curl}(\operatorname{div}\pi_\nu) \\ &-(1/eNB^2)(\boldsymbol{B}\times\operatorname{div}\pi_{i0})\cdot\nabla n_\nu - \operatorname{div}\left\{\frac{N}{\omega_\nu B}\left[\nabla_\perp\frac{\partial\tilde{\phi}}{\partial t} + \frac{\partial}{\partial t}\left(\frac{1}{n_\nu q_\nu}\operatorname{div}\pi_\nu\right)_\perp\right]\right\} \\ &-\operatorname{div}\left\{\frac{N}{\omega_\nu^2}\left[\frac{\partial^2 \boldsymbol{v}_{\nu\perp}}{\partial t^2} - 2\left(\frac{\partial \boldsymbol{v}_\nu}{\partial t}\times\boldsymbol{\Omega}\right)_\perp\right]\right\}\end{aligned} \tag{8.9}$$

$$\tilde{n}_i - \tilde{n}_e = -(\varepsilon_0/e)\nabla^2\tilde{\phi} \tag{8.10}$$

$$\tilde{p}_\nu = (5P_\nu/3N)\tilde{n}_\nu \tag{8.43}$$

$$\boldsymbol{u}_{B\nu} = (10P_\nu/3Nq_\nu B^3)\boldsymbol{B} \times \nabla B \tag{8.46}$$

$$m\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\boldsymbol{w}) = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{P}) = q(\boldsymbol{E} + \boldsymbol{w}\times\boldsymbol{B}) \tag{9.28}$$

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t} = q\boldsymbol{w}\cdot\boldsymbol{E}; \quad \mathscr{E} = \gamma mc^2 \tag{9.29}$$

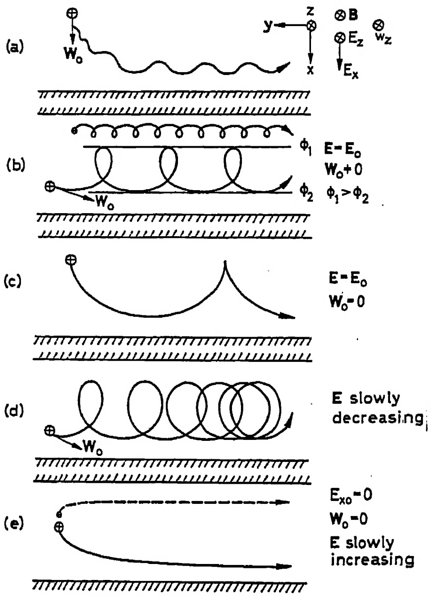

Fig. 2. 4. Orbits of charged particle in a homogeneous magnetostatic field $\boldsymbol{B}$ and a homogeneous, time-dependent electric field $\boldsymbol{E}$. a. Definition of coordinates. b. Constant electric field $\boldsymbol{E}_0$. The electric drift is the same for particles of both signs. c. Constant electric field and zero initial velocity $\boldsymbol{W}_0$. d. Electric field slowly decreasing. The rate of change of the electric field drift is shown on a strongly exaggerated scale. e. Electric field slowly increasing from zero. No initial velocity. Displacements in x direction strongly exaggerated.

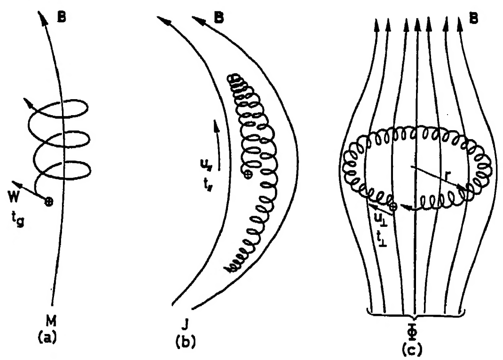

Fig. 4.1. Periodic motions of a charged particle in a magnetic field. a. Gyration at velocity $\boldsymbol{W}$ around the field lines and with the gyro period t_g. b. Oscillations between magnetic mirrors at velocity $u_{\|}$ along the field lines and with the period $t_{\|}$. c. Repeated drift around the whole configuration at velocity $u_\perp$ across the field lines and with the period $t_\perp$.

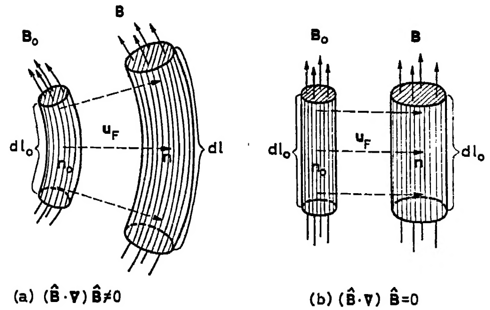

Fig. 6.2. Expansion of ionized matter moving across an inhomogeneous magnetic field. No longitudinal motion is taking place ($v_\parallel = 0$). (a) Curved field lines. (b) Straight field lines.

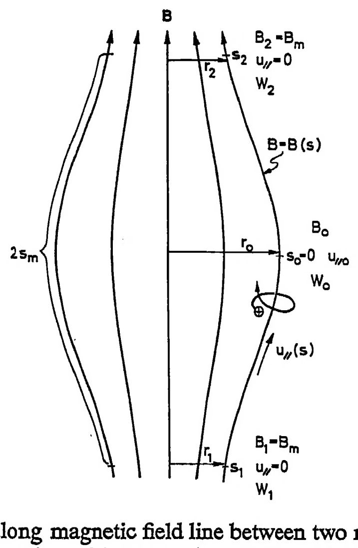

Fig. 6.3. Particle moving along magnetic field line between two mirror points at s_1 and s_2. The cross section with the weakest field B_0 is given by r_0.

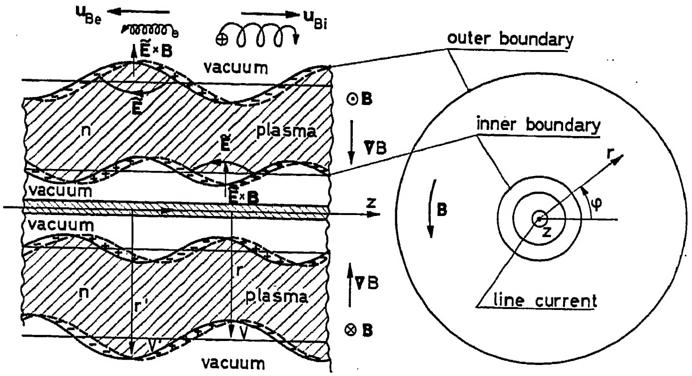

Fig. 8.2. Cylindrical shell of plasma confined in the magnetic field from an axial line current.

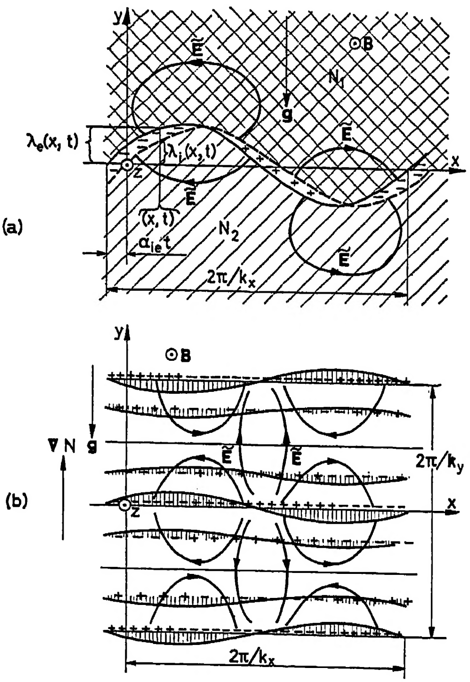

Fig. 8.5. Charge separation in the interior of a magnetized plasma due to gravitation drifts of density perturbations. (a) Sharp boundary between two regions of densities N_1 and N_2. (b) Continuous unperturbed density gradient, dN/dy, with sinusoidal density perturbations in the x and y directions forms a "bumpy" pattern of surfaces of constant density. The vertically shaded areas indicate excess of density when deviation from horizontal line is downwards and deficit of density when it is upwards.